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Well-Posedness and Finite Element Approximations for Elliptic SPDEs with Gaussian Noises

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Abstract. The paper studies the well-posedness and optimal error estimates of spectral finite element approximations for the boundary value problems of semi-linear elliptic SPDEs driven by white or colored Gaussian noises. The noise term is approximated through the spectral projection of the covariance operator, which is not required to be commutative with the Laplacian operator. Through the convergence analysis of SPDEs with the noise terms replaced by the projected noises, the well-posedness of the SPDE is established under certain covariance operator-dependent conditions. These SPDEs with projected noises are then numerically approximated with the finite element method. A general error estimate framework is established for the finite element approximations. Based on this framework, optimal error estimates of finite element approximations for elliptic SPDEs driven by power-law noises are obtained. It is shown that with the proposed approach, convergence order of white noise driven SPDEs is improved by half for one-dimensional problems, and by an infinitesimal factor for higher-dimensional problems.

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Key words: Elliptic stochastic partial differential equation, spectral approximations, finite element approximations, power-law noise.

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1 Introduction

In recent years, random disturbance as a form of uncertainty has been increasingly treated as an essential modeling factor in the analysis of complex phenomena. Adding such uncertainty to partial differential equations (PDEs) which model such physical and engineering phenomena, one derives stochastic PDEs (SPDEs) as improved mathematical modeling tools. SPDEs derived from fluid flows and other engineering fields are often assumed to be driven by white noises that have constant power spectral densities [8]. However, most random fluctuations in complex systems are correlated acting on different frequencies in which case the noises are called colored noises [10].

Elliptic SPDEs driven by white noises and colored noises have been considered by many authors, see e.g. [1,5,6,15,16] for white noises, [11,12,15] for colored noises determined by Riesz-type kernels, [3, 4] for fractional noises, and [14] for power-law noises.

The main objective of this study is to investigate the well-posedness and optimal error estimate of spectral finite element approximations for the following semilinear elliptic SPDE:

$$\begin{aligned} &-\Delta u(x,\xi)=f(u(x,\xi))+\dot{W}^Q(x,\xi), && x\in\mathscr{O},\ \xi\in\Omega,\\ &u(x,\xi)=0, && x\in\partial\mathscr{O},\ \xi\in\Omega. \end{aligned} \tag{1.1}$$

Here $(\Omega,\mathscr{F},\mathbb{P})$ is a probability space and $\mathscr{O}\subset\mathbb{R}^d$ is a bounded domain with regular boundary $\partial\mathscr{O}$, $f:\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous function, and $\dot{W}^Q$ is a class of centered Gaussian noises with covariance operator Q.

The existence of the unique solution for SPDE (1.1) driven by the white noise, i.e., $Q=I$, has been established in [2] by converting the problem into an integral equation. In this paper, we establish a covariance operator-dependent condition for the well-posedness of SPDE (1.1) through the convergence analysis for a sequence of solutions of SPDEs with the noise term in SPDE (1.1) replaced by its spectral projections. This sequence of SPDEs will play an important role in constructing our numerical solutions for SPDE (1.1).

To obtain numerical solutions, we apply the finite element method to the aforementioned SPDEs whose noises are the spectral projections of the original noise. In previous studies [1, 6], the noises are approximated by piecewise constants in space. In addition, it is required that the eigenvectors of the Laplacian also diagonalize the covariance operator of the noise, i.e., the Laplacian operator and the covariance operators are commutative. In this study, the commutative assumption is no longer required. Another improvement over the results of [1,6]