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Staggered Lagrangian Discretization Based on Cell-Centered Riemann Solver and Associated Hydrodynamics Scheme

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Abstract. The aim of the present work is to develop a general formalism to derive staggered discretizations for Lagrangian hydrodynamics on two-dimensional unstructured grids. To this end, we make use of the compatible discretization that has been initially introduced by E. J. Caramana et al., in J. Comput. Phys., 146 (1998). Namely, momentum equation is discretized by means of subcell forces and specific internal energy equation is obtained using total energy conservation. The main contribution of this work lies in the fact that the subcell force is derived invoking Galilean invariance and thermodynamic consistency. That is, we deduce a general form of the sub-cell force so that a cell entropy inequality is satisfied. The subcell force writes as a pressure contribution plus a tensorial viscous contribution which is proportional to the difference between the nodal velocity and the cell-centered velocity. This cell-centered velocity is a supplementary degree of freedom that is solved by means of a cell-centered approximate Riemann solver. To satisfy the second law of thermodynamics, the local subcell tensor involved in the viscous part of the subcell force must be symmetric positive definite. This subcell tensor is the cornerstone of the scheme. One particular expression of this tensor is given. A high-order extension of this discretization is provided. Numerical tests are presented in order to assess the efficiency of this approach. The results obtained for various representative configurations of one and two-dimensional compressible fluid flows show the robustness and the accuracy of this scheme.

AMS subject classifications: 52B10, 65D18, 68U05, 68U07

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1 Introduction

In Lagrangian hydrodynamics methods, a computational cell moves with the flow velocity. In practice, this means that the cell vertices move with a computed velocity, the cell faces being uniquely specified by the vertex positions. Thus, Lagrangian methods can capture contact discontinuity sharply in multi-material fluid flows. However, in the Lagrangian framework, one has to discretize not only the gas dynamics equations but also the vertex motion in order to move the mesh. Moreover, the numerical fluxes of the physical conservation laws must be determined in a compatible way with the vertex velocity so that the geometric conservation law (GCL) is satisfied, namely the rate of change of a Lagrangian volume has to be computed coherently with the node motion. This critical requirement is the cornerstone of any Lagrangian multi-dimensional scheme.

The most natural way to solve this problem employs a staggered discretization in which position, velocity and kinetic energy are centered at points, while thermodynamic variables (density, pressure and specific internal energy) are defined within cells. The dissipation of kinetic energy into internal energy through shock waves is ensured by an artificial viscosity term. Since the seminal works of von Neumann and Richtmyer [33], and Wilkins [34], many developments have been made in order to improve the accuracy and the robustness of staggered hydrodynamics [8, 11, 12]. More specifically, the construction of a compatible staggered discretization leads to a scheme that conserves total energy in a rigorous manner [9, 10].

An alternative to the previous discretizations is to derive a Lagrangian scheme based on the Godunov method [18]. In the Godunov-type method approach, all conserved quantities, including momentum, and hence cell velocity, are cell-centered. The cell-face quantities, including a face-normal component of the velocity, are available from the solution of an approximate Riemann problem at each cell face. However, it remains to determine the vertex velocity in order to move the mesh. In the early work [1] the flux computation was not compatible with the node displacement, and hence the GCL was not satisfied. This incompatibility generated additional spurious components in the vertex velocity field whose correction required expensive treatment [17]. An important achievement concerning the compatibility between flux discretization and vertex velocity computation has been introduced in [15,27]. In these papers, the authors present schemes in which the interface fluxes and the node velocity are computed coherently thanks to an approximate Riemann solver located at the nodes. This original approach leads to first-order conservative schemes which satisfy a local semi-discrete entropy inequality. The multi-dimensional high-order extension of these schemes are developed in [13,25,26,28].

The staggered discretization of variables (kinematic variables located at nodes, thermodynamic ones at cell centers) allows the scheme to fulfill naturally the GCL compatibility requirement and at the same time to construct a discrete divergence operator. The discretizations of momentum and specific internal energy are derived from each other by use of the important concepts of subcell mass, subcell force and total energy conservation [10]. This compatible hydrodynamics algorithm is thus designed to conserve