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A Two-Scale Asymptotic Analysis of a Time-Harmonic Scattering Problem with a Multi Layered Thin Periodic Domain

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Abstract. The scope of this paper is to show how a two-scale asymptotic analysis, based on a superposition principle, allows us to derive high order approximate boundary conditions for a scattering problem of a time-harmonic wave by a thin and tangentially periodic multi-layered domain. The periods are assumed of the same order of the thickness. New terms like memory effect and variance-covariance ones are observed contrarily to the laminar case. As a result, optimal error estimates are obtained.

AMS subject classifications: 78A45, 35B27, 35B25, 35C20

Key words: Two-scale asymptotic analysis, superposition principle, tangential periodicity, thin layer, approximate boundary condition, time-harmonic scattering.

1 Introduction

In industrial word, a wide variety of materials are coated or connected by a thin structure. For example, electronic devices, patch antennas, radar absorbing paints, self-focusing lens are some illustrations of this situation. Many authors have been devoted to solve the problem of the coating effect by developing robust methods for approximating the solution inside the thin layer, see [3, 8, 9, 16, 17, 25–27] and the references therein. Their main approach consists of constructing an equivalent boundary (or transmission) condition which is able to memorize the effect of the thin shell in an approximate way. Our motivation in this paper is to show how this memory effect can be captured in the case of the scattering of a time-harmonic wave by an obstacle coated by a multi layered thin periodic domain, the periods are small of same order of the thickness. More precisely, besides the non commutativity of a two-step procedure, i.e., homogenization for a fixed

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thickness δ followed by an asymptotic analysis for small δ, or vice-versa, it is shown that neither the one neither the other is able to give an answer in our case. So, inspired from the two-scale convergence technique [7, 20, 21] which takes full advantage of the periodicity information, a suitable superposition of test functions oscillating at same order of δ is used to derive correctly some variance-covariance terms. The idea is similar to the one used in [24] for rough surfaces when small details are not visible within a standard homogenization technique. Nevertheless, such an approach leads to a loss of a half power of δ in the rate of convergence when compared with the case of an homogeneous thin layer. This loss is due essentially to a compensation rule that keeps traces of some lower order terms in the proof of the convergence theorem (cf., e.g., [3]). Finally, it is shown how to optimize it by the use of the simple but clever trick (cf., e.g., [28]) making it possible to obtain optimal estimates from non-optimal ones and the existence of the ansatz up to next order only.

In Section 2, a brief description of the model is presented for a 2D situation. In Section 3, a two-scale asymptotic analysis (cf., e.g., [3, 10, 13, 21, 24]) with respect to the thickness and the period permits to justify the terms in the periodic ansatz proposed and a first convergence theorem is obtained for the Neumann case. In Section 4, apparently new to our knowledge, some approximate boundary conditions are derived until the second order which makes the difference significative with respect to the homogeneous or even the laminar cases (cf., e.g., [9, 29]). Mainly, a convergence result is proved and it is shown how to optimize it providing more regularity on the data.

2 The model setting

In all what follows, standards tools from the functional analysis of PDE(s) and differential geometry background are used without comments (cf., e.g., [11, 12, 15]). Let $\Omega_{\delta,\infty}$ be an exterior domain in $\mathbb{R}^2$ with boundary Γ_δ (compact C^∞ manifold) such that $\Omega_{\delta,\infty} = \Omega_\delta^+ \cup \Gamma \cup \Omega_\infty$. Γ is an interface parallel to Γ_δ and δ is a non-negative small parameter. $\bar{\Omega}_\delta^+ = \{x \in \Omega_{\delta,\infty} : d(x,\Gamma) \le \delta\}$ represents the thin layer of thickness δ and Ω_∞ is the exterior domain to the coated scatterer. Let $f \in L^2(\mathbb{R}^2)$ compactly supported in Ω_∞. From now on, v^+ (respectively v^-) will denote the restriction of a distribution v defined on $\Omega_{\delta,\infty}$ to the subset Ω_δ^+ (respectively Ω_∞). The problem is to find a complex valued distribution u_δ solution of the scattering problem:

$$\Delta u_\delta^- + k^2 u_\delta^- = -f, \tag{2.1}$$

$$div\left(\alpha_\delta \nabla u_\delta^+\right) + k^2 \beta_\delta u_\delta^+ = 0, \tag{2.2}$$

$$u_\delta^+|_\Gamma = u_\delta^-|_\Gamma, \tag{2.3}$$

$$\alpha_\delta \partial_{\mathbf{n}} u_\delta^+|_\Gamma = \partial_{\mathbf{n}} u_\delta^-|_\Gamma, \tag{2.4}$$

$$\lim_{|x|\to\infty} |x| \left(\nabla u_\delta^- . \frac{x}{|x|} - iku_\delta^- \right) = 0, \tag{2.5}$$