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# A Fast Local Level Set Method for Inverse Gravimetry

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**Abstract.** We propose a fast local level set method for the inverse problem of gravimetry. The theoretical foundation for our approach is based on the following uniqueness result: if an open set $D$ is star-shaped or $x_3$-convex with respect to its center of gravity, then its exterior potential uniquely determines the open set $D$. To achieve this purpose constructively, the first challenge is how to parametrize this open set $D$ as its boundary may have a variety of possible shapes. To describe those different shapes we propose to use a level-set function to parametrize the unknown boundary of this open set. The second challenge is how to deal with the issue of partial data as gravimetric measurements are only made on a part of a given reference domain $\Omega$. To overcome this difficulty, we propose a linear numerical continuation approach based on the single layer representation to find potentials on the boundary of some artificial domain containing the unknown set $D$. The third challenge is how to speed up the level set inversion process. Based on some features of the underlying inverse gravimetry problem such as the potential density being constant inside the unknown domain, we propose a novel numerical approach which is able to take advantage of these features so that the computational speed is accelerated by an order of magnitude. We carry out numerical experiments for both two- and three-dimensional cases to demonstrate the effectiveness of the new algorithm.

**AMS subject classifications**: 52B10, 65D18, 68U05, 68U07

**Key words**: Level set methods, inverse gravimetry, fast algorithms, numerical continuation.

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# 1 Introduction

Let $\Omega$ be a domain in $\mathcal{R}^n$ with connected $\mathcal{R}^n\backslash\bar{\Omega}$ and let $U(\cdot;\mu)$ be the potential of a measure $\mu$ with respect to the kernel for the Laplacian operator. The inverse problem of potential theory is formulated as follows [8]: find a measure $\mu$ with support contained in $\Omega$ from its potential $U(\cdot;\mu)$ in $\mathcal{R}^n\backslash\Omega$. As stated, the solution to this problem is notoriously non unique. In gravimetry it is reasonable to assume that the measure $\mu$ is a volume mass distribution with density $f$ on an open set $D$ (an open bounded subset of $\Omega$); moreover, in many realistic situations $f$ is a known constant. Then, under some geometrical assumptions on the open set $D$ one can claim its uniqueness for the inverse problem.

As known, the inverse gravimetry problem is severely ill-posed [8], and many regularization techniques have been proposed to obtain a conditionally well-posed problem so as to solve this important inverse problem numerically. Most recently, the inverse gravimetry problem with an unknown density $f$ have been tackled in [1] by using the total-variation regularization. Although the numerical reconstruction in [1] is successful for some examples, the proposed methods in [1] may run into difficulties due to the presence of many local minima of the to-be-minimized non convex functional. We propose to tackle the inverse gravimetry problem with density $f=1$ on an unknown open set $D$ by using the level set method. The theoretical foundation for our approach is the uniqueness result presented in [8]: namely, if $D$ is star-shaped with respect to its center of gravity or $x_3$-convex, then the exterior potential uniquely determines $D$. To achieve our purpose, the first challenge is how to represent $D$. We propose to consider $\partial D$ as the zero-level set of a level-set function to be found. The level set method [13] is a powerful approach for interface or shape-optimization problems, which can take care of interface merging and topological changes automatically. The second challenge is how to deal with the issue of partial data as the potential measurement is only made on a part of the whole boundary of the computational domain. To overcome this difficulty, inspired by ideas in [6] we propose a numerical continuation approach to obtain fictitious measurements of the potential due to the unknown domain. The third challenge is how to speed up the level set inversion process. Based on the features of the underlying inverse gravimetry problem, we propose a novel approach to speed up the computational process by an order of magnitude, so that we can carry out numerical experiments for both two- and three-dimensional cases.

## 1.1 Related work

For inverse (obstacle) problems the level set method has been first used by Santosa [16]. Later on, there are many efforts to analyze this method and extend this beautiful idea to a variety of inverse problems; see [2,3,7,11,19] and references therein.

The closest to our work is the recent interesting paper [18] in which a method based on multiple level sets has been proposed for piecewise constant volume density recon-