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Numerical Investigation on the Boundary Conditions for the Multiscale Base Functions

Shan Jiang[1] and Yunqing Huang[2,*]

[1] *Department of Mathematics, Xiangtan University, Xiangtan 411105, China.*
[2] *Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Institute for Computational and Applied Mathematics, Xiangtan University, Xiangtan 411105, China.*

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Abstract. We study the multiscale finite element method for solving multiscale elliptic problems with highly oscillating coefficients, which is designed to accurately capture the large scale behaviors of the solution without resolving the small scale characters. The key idea is to construct the multiscale base functions in the local partial differential equation with proper boundary conditions. The boundary conditions are chosen to extract more accurate boundary information in the local problem. We consider periodic and non-periodic coefficients with linear and oscillatory boundary conditions for the base functions. Numerical examples will be provided to demonstrate the effectiveness of the proposed multiscale finite element method.

AMS subject classifications: 35B30, 35J25, 65C20, 65N12, 65N30

Key words: Multiscale finite element method, multiscale base functions, oscillatory boundary condition, periodic coefficient, non-periodic coefficient.

1 Introduction

Many multiscale problems are often described by partial differential equations (PDEs) with highly oscillating coefficients. In practice, the coefficients may contain many scales spanning over a great extent [3]. On one hand, the direct use of traditional numerical methods, such as standard finite element method (FEM) or finite difference method (FDM), to the multiscale problems is very difficult since the mesh size has to be extremely small. On the other hand, the main interest is to acquire the large scale solution with accuracy instead of finding the small scale characters in detail. The multiscale finite element

*Corresponding author. *Email addresses:* `shanjiang1980@gmail.com` (S. Jiang), `huangyq@xtu.edu.cn` (Y. Huang)

method (MFEM), whose goal is to obtain the large scale solution accurately and efficiently, is to capture large scale information by constructing the multiscale finite element base functions. This can be achieved by solving the base functions from the local problem in the elements. With proper boundary conditions, the base functions are adaptive to the features of the differential operator.

To capture the large scale solutions without resolving the small scale details, Babuška & Osborn [2] (for one-dimensional problems) and Babuška et al. [1] (for special two-dimensional problems) presented the generalized finite element method by introducing modified base functions that are based on the differential operator. Hou & Wu [12] extended the idea of [1, 2] and proposed the multiscale finite element method by solving the local homogenization problems for the base functions. Hou et al. [13] and Efendiev & Wu [8] provided many theoretical analysis and numerical experiments for the MFEM. Engquist & Luo [9] studied the convergence of the multigrid method for highly oscillatory elliptic problems on a new coarse-grid finite difference scheme. Huang & Xu [14, 15] applied the partition of unity method (PUM) to the multiscale problems with highly oscillating coefficients, and proved that the PUM admitted optimal convergence rate with nonmatching and overlapping grids. In [4], Chen & Cui constructed a special multiscale rectangular element space whose base functions consisting of bilinear functions and bubble-like functions. In [5], Chen & Hou proposed a mixed multiscale finite element method with an over-sampling technique, which solves the local Neumann boundary value problem for the bases. Chen & Yue [6] considered the oversampling multiscale finite element method with a new upscaling technique for resolving the well singularities. Jenny et al. [16] and He & Ren [11] applied the multiscale finite volume method in subsurface flow simulation and for solving the ground-water flow problems, respectively. Ren & E [19] and Yue & E [20] studied the heterogeneous multiscale method for the modeling of complex fluids with application to two-phase porous media flow. In [17], Ming & Yue presented an overview of the recent development on the multiscale numerical methods. Efendiev & Hou [7] discussed the applications of the MFEM to two-phase immiscible flow simulation in which limited global information is taken into account, and the applications to inverse problems are also discussed. Nassehi et al. [18] developed the MFEM using bubble functions thus obtained stable solutions without excessive mesh refinement near the wall. In [10], a systematic review to the heterogeneous multiscale method (HMM), including the fundamental designing philosophy and the error analysis, is presented. Yue & E [21] systematically investigated the issues in the multiscale modeling, and discussed the mixed Dirichlet-Neumann boundary condition in porous media.

An advantage of the multiscale finite element method is that it can reduce the size of computation. For example, let N be the number of elements in each spatial direction, and let M be the number of subcell elements in each direction for solving the base functions. Then there are a total of $(MN)^d$ (d is the dimension) elements at the fine grid level. For the FEM, the computer memory required to solve the problem at the fine grid is $\mathcal{O}(M^dN^d)$, in contrast with the MFEM which requires only $\mathcal{O}(M^d+N^d)$ amount of memory. Moreover,