Commun. Comput. Phys.
doi: 10.4208/cicp.250909.210111a

Vol. **10**, No. 5, pp. 1132-1160
November 2011

Numerical Entropy and Adaptivity for Finite Volume Schemes

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Received 25 September 2009; Accepted (in revised version) 21 January 2011

Communicated by Chi-Wang Shu

Available online 2 August 2011

Abstract. We propose an a-posteriori error/smoothness indicator for standard semi-discrete finite volume schemes for systems of conservation laws, based on the numerical production of entropy. This idea extends previous work by the first author limited to central finite volume schemes on staggered grids. We prove that the indicator converges to zero with the same rate of the error of the underlying numerical scheme on smooth flows under grid refinement. We construct and test an adaptive scheme for systems of equations in which the mesh is driven by the entropy indicator. The adaptive scheme uses a single nonuniform grid with a variable timestep. We show how to implement a second order scheme on such a space-time non uniform grid, preserving accuracy and conservation properties. We also give an example of a p-adaptive strategy.

AMS subject classifications: 65M08, 65M50, 76M12

Key words: Finite volume schemes, hyperbolic systems, local grid refinement, entropy.

1 Introduction

The a-posteriori error indicator proposed in this paper can be constructed for multidimensional systems of conservation laws. However, for simplicity, in the following we will mainly consider one-dimensional hyperbolic systems of equations of the form:

$$u_t + f_x(u) = 0. \tag{1.1}$$

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Here $u(t,x)$ is a function from $\mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^m$, where m is the number of equations of the system, f is the flux function, and we assume that f is a smooth function from $\mathbb{R}^m \rightarrow \mathbb{R}^m$. We suppose that the system is strictly hyperbolic, i.e., that the Jacobian $A(u) = J(f)$ has m real eigenvalues and a complete set of corresponding eigenvectors.

It is well known that solutions of initial value problems for (1.1) may loose their regularity in a finite time even if the initial data are smooth, developing shock waves. In this case, the solution must be understood in the weak sense, and uniqueness of the solution is lost. In order to retrieve uniqueness, the system must be completed with an entropy inequality, characterizing the unique admissible weak solutions of (1.1). Thus, we will consider systems of the form (1.1) possessing an entropy-entropy flux pair, that is we assume there exist a convex function $\eta(u)$ and a corresponding entropy flux $\psi(u)$ verifying the compatibility condition $\nabla^T \eta A(u) = \nabla^T \psi$ (see [8]). Then it is well known that entropy solutions of (1.1) must satisfy the entropy inequality

$$\eta_t + \psi_x(u) \leq 0 \tag{1.2}$$

in the weak sense for all entropies.

Numerical integration of (1.1) is challenging because the solution can exhibit a very complex structure: discontinuities may arise and disappear through the interaction with other waves present in the flow. For these reasons, several attempts to the construction of adaptive grids have appeared in the literature. Adaptive grids seek to achieve a good resolution in regions where the flow varies rapidly, and an effective error control where the flow does not have a complex structure, and high resolution is not needed.

High resolution can be obtained using a fine grid, and/or using a high order scheme. In the first case, the CPU time increases rapidly, because the CFL stability condition imposes an upper bound on the grid ratio $\lambda = \Delta t / h$, where Δt is the time step and h is the grid spacing, so that a fine grid requires a small time step. In the second case, the presence of discontinuities may result in non-linear instabilities in a high order numerical solution, so that the scheme itself must become non linear to prevent the onset of spurious oscillations. There is a huge literature on non-oscillatory high order schemes for hyperbolic problems. Here we just mention the reference [18] especially for second order schemes, and [28] for higher order schemes. As the order increases, these schemes become more costly, not only because their structure and the mechanisms designed to prevent an oscillatory behavior become more complex, but also because usually they seek non oscillatory stencils to compute the solution, and this requires the use of characteristic variables to prevent the selection of stencils containing wave interactions, see [27].

Finally, finite volume schemes usually give a good resolution of shocks even on coarse grids, because the smearing effect of numerical viscosity is counterbalanced by the steepening mechanism of converging characteristics. This does not occur on contact discontinuities, along which the characteristic fields are parallel, and only numerical diffusion is active [11].

Thus an effective adaptive algorithm must be driven by an indicator which should be able not only to provide a robust a posteriori measure of the local error, but should also