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An Energy Absorbing Far-Field Boundary Condition for the Elastic Wave Equation

N. Anders Petersson* and Björn Sjögreen

Center for Applied and Scientific Computing L-550, Lawrence Livermore National Laboratory, Livermore, CA 94551, USA.

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Abstract. We present an energy absorbing non-reflecting boundary condition of Clayton-Engquist type for the elastic wave equation together with a discretization which is stable for any ratio of compressional to shear wave speed. We prove stability for a second-order accurate finite-difference discretization of the elastic wave equation in three space dimensions together with a discretization of the proposed non-reflecting boundary condition. The stability proof is based on a discrete energy estimate and is valid for heterogeneous materials. The proof includes all six boundaries of the computational domain where special discretizations are needed at the edges and corners. The stability proof holds also when a free surface boundary condition is imposed on some sides of the computational domain.

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Key words: Elastic wave equation, far-field boundary condition, finite differences, stability, energy estimate.

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1 Introduction

In regional simulations of seismic wave propagation, the extent of the computational domain must be limited to make the problem computationally tractable. Some form of far-field absorbing boundary condition needs to be imposed where the computational domain is truncated such that waves can propagate out of the computational domain without being reflected due to the artificial boundary. For a material with constant wave speeds, and a domain with a single planar boundary, it is possible to derive a boundary condition which allows all waves to exit the domain without any artificial reflection. However, such a boundary condition involves a pseudo-differential operator and is therefore non-local in space and unsuitable for numerical computations.

*Corresponding author. *Email addresses:* `andersp@llnl.gov` (N. A. Petersson), `sjogreen2@llnl.gov` (B. Sjögreen)

One of the first practically useful far-field boundary condition for the elastic wave equation was derived by Clayton and Engquist [4], where the authors presented a hierarchy of boundary conditions by approximating the exact pseudo-differential operator to increasing order of accuracy in the angle of incidence. (All boundary conditions in the hierarchy are perfectly non-reflecting for waves of normal incidence.) A slightly different approach was suggested by Higdon in [9], where the boundary condition is obtained by component wise application of a scalar non-reflecting boundary condition. Higdon also derived a hierarchy of boundary conditions with increasingly absorbing properties. In the case of a scalar wave equation, the Higdon and Clayton-Engquist boundary conditions are equivalent. First order Clayton-Engquist conditions have been used extensively in large scale computations of seismic wave propagation, see [5]. However, instabilities have been reported for the third order condition for some values of the wave speeds [12].

The perfectly matched layer (PML) is a more modern boundary condition which was originally developed for Maxwell's equations by Berenger [2] and has been studied in numerous subsequent papers, see for example [16] and the references therein. Perfectly matched layers have superior non-reflecting properties compared to low order Clayton-Engquist or Higdon conditions, but they are also more complicated to implement and require correct tuning of the size and strength of the absorbing layer. PMLs for the elastic wave equation were developed in [1,10]. Unfortunately, the PML boundary condition can become unstable when it interacts with surface waves along material discontinuities [17].

Higdon [8] performed a normal-mode stability analysis for a class of discretized non-reflecting boundary conditions for the elastic wave equation, which includes the first order Clayton-Engquist condition as a special case. In particular, Higdon showed stability for a first order accurate discretization of the Clayton-Engquist condition. Note that the normal mode analysis is only valid for half-space problems with homogeneous materials and does not take corners or edges into account. Furthermore, the stability concept in the normal mode analysis only guarantees the solution to be bounded independently of the grid size for a fixed, finite, interval in time. It does not exclude the possibility that the solution may grow as the time interval is made longer. We remark that the discretization given in the original paper by Clayton and Engquist [4] is second order accurate and is therefore not covered by Higdon's analysis.

In seismic simulations, the material properties are not known very precisely and there are often uncertainties associated with the source terms modeling the spatial distribution and temporal variation of the slip during an earthquake. We therefore believe that in many realistic seismic simulations, adequate accuracy can be obtained by using low order outflow boundary conditions as long as they are stable. Often the material properties vary rapidly on the computational grid and this can cause stability problems for the Clayton-Engquist conditions, which are derived under the assumption of constant coefficients. Additional stability problems occur for large ratios between the compressional and shear wave speeds: c_p/c_s. Here,

$$c_p = \sqrt{(2\mu+\lambda)/\rho}, \quad c_s = \sqrt{\mu/\rho},$$