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# A Preconditioned Recycling GMRES Solver for Stochastic Helmholtz Problems

Chao Jin[1] and Xiao-Chuan Cai[2,*]

[1] *Cadence Design Systems, 555 River Oaks Parkway, San Jose, CA 95134, USA.*
[2] *Department of Computer Science, University of Colorado at Boulder, Boulder, CO 80309, USA.*

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**Abstract.** We present a parallel Schwarz type domain decomposition preconditioned recycling Krylov subspace method for the numerical solution of stochastic indefinite elliptic equations with two random coefficients. Karhunen-Loève expansions are used to represent the stochastic variables and the stochastic Galerkin method with double orthogonal polynomials is used to derive a sequence of uncoupled deterministic equations. We show numerically that the Schwarz preconditioned recycling GMRES method is an effective technique for solving the entire family of linear systems and, in particular, the use of recycled Krylov subspaces is the key element of this successful approach.

**AMS subject classifications**: 35R60, 60H15, 60H35, 65C30, 47B80, 65N55, 65M55

**Key words**: Recycling GMRES, domain decomposition, additive Schwarz preconditioner, stochastic Helmholtz equation.

## 1 Introduction

Tremendous progress has been made recently in developing reliable and fast algorithms for solving partial differential equations with uncertainty in the coefficients [1, 5–7, 10, 15, 21, 22]. We study a domain decomposition preconditioned recycling Krylov subspace technique [18] for solving some stochastic partial differential equations. In particular, we focus on a class of indefinite elliptic equations which are more sensitive to the stochastic perturbations. The method was introduced in [14] for solving the uncoupled systems of equations arising from the discretization of stochastic elliptic equations with a single

*Corresponding author. *Email addresses:* chao@cadence.com (C. Jin), cai@cs.colorado.edu (X.-C. Cai)

 

random coefficient. In this paper, we extend the approach to a class of indefinite elliptic problems with random diffusion and reaction coefficients

$$\begin{cases} -\nabla\cdot(a(x,\omega_1)\nabla u(x,\omega))-c(x,\omega_2)u(x,\omega)=f(x) & x\in D,\ \omega_i\in\Omega,\\ u(x,\omega)=0 & x\in\partial D,\ \omega_i\in\Omega, \end{cases}$$

where

$$\omega=(\omega_1,\omega_2),\quad a(x,\omega_1)\geq\alpha>0,\quad c(x,\omega_2)\geq 0,$$

$D$ is the domain for $x$, and $\Omega$ is the sample space for $\omega_i, i=1,2$. This type of Helmholtz equations appears in many important applications such as computational acoustics and is rather difficult to solve by iterative methods because of the existence of both positive and negative eigenvalues and eigenvalues that are very close to zero. Slight perturbation of the coefficients of the equations may move some of eigenvalues from positive to negative or to somewhere very close to zero. Through a large number of numerical experiments, we found that traditional preconditioning technique, which uses one matrix in the sequence to precondition another matrix in the same sequence, is not very effective. The precise reason is not clear, but it may be because the eigen bounds (for both positive and negative eigenvalues) are too different for the un-preconditioned matrix and the preconditioning matrix. On the other hand, our numerical experiments show that the recycling Krylov subspace method is very effective in this situation. We believe this is due to the fact that the recycling Krylov subspace method provides a very good initial guess for solving the next system. We mention that the idea of re-using preconditioner and the idea of re-using the Krylov subspace are not new, but to use the combination for solving this type of indefinite problems is new and the observation of the effectiveness of the method has never been reported elsewhere.

There are several approaches for solving the problems, we follow [1, 10] to use the so-called double orthogonal basis to decouple the high dimensional equation in the probability space and produce a sequence of independent systems

$$A_i x_i = b_i,\quad i=1,2,\cdots, \tag{1.1}$$

where the matrices $A_i$ and right-hand sides $b_i$ are somewhat related but independent from each other. Each system in the sequence is indefinite, and is rather difficult to solve using any iterative methods. We use the recently introduced recycling Krylov subspace method [18], which starts the iteration from a Krylov subspace created from a previous system. For preconditioning, we use an overlapping additive Schwarz domain decomposition method [20]. Our parallel implementation is based on the Portable Extensible Toolkit for Scientific computation (PETSc) package from Argonne National Laboratory [2].

The rest of the paper is organized as follows. In Section 2, we describe the stochastic Galerkin method including the stochastic weak formulation, the Karhunen-Loève (KL) expansion, the double orthogonal basis, and the discretization. Section 3 presents the additive Schwarz preconditioned recycling Krylov subspace method. Some experimental results are reported in Section 4.