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The Recursive Formulation of Particular Solutions for Some Elliptic PDEs with Polynomial Source Functions

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Abstract. In this paper we develop an efficient meshless method for solving inhomogeneous elliptic partial differential equations. We first approximate the source function by Chebyshev polynomials. We then focus on how to find a polynomial particular solution when the source function is a polynomial. Through the principle of the method of undetermined coefficients and a proper arrangement of the terms for the polynomial particular solution to be determined, the coefficients of the particular solution satisfy a triangular system of linear algebraic equations. Explicit recursive formulas for the coefficients of the particular solutions are derived for different types of elliptic PDEs. The method is further incorporated into the method of fundamental solutions for solving inhomogeneous elliptic PDEs. Numerical results show that our approach is efficient and accurate.

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1 Introduction

We consider the boundary value problem,

$$\begin{aligned} Lu(x,y) &= f(x,y), \ (x,y)\in\Omega, \\ Bu(x,y) &= g(x,y), \ (x,y)\in\partial\Omega, \end{aligned} \tag{1.1}$$

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where $\Omega \subset R^2$ is a simply connected domain whose boundary is a simple closed curve $\partial\Omega$, L and B are the differential operators on u over the interior of Ω and the boundary $\partial\Omega$ respectively. We assume that the operator L is of elliptic type. Efficient and accurate solution techniques for the elliptic boundary value problem can easily find applications in diverse problems in mechanics, gravitation, electricity, and magnetism.

In the framework of boundary methods, a widely used approach is to split the solution of Problem (1.1) into a particular solution $u_p(x,y)$ that satisfies

$$Lu_p(x,y) = f(x,y), \ (x,y) \in \Omega, \tag{1.2}$$

and its associated homogeneous solution $u_h(x,y)$ that satisfies

$$\begin{aligned} &Lu_h(x,y) = 0, \ (x,y) \in \Omega, \\ &Bu_h(x,y) = g(x,y) - Bu_p(x,y), \ (x,y) \in \partial\Omega. \end{aligned} \tag{1.3}$$

Once a particular solution u_p is known, the influence on the solution by the inhomogeneous term f has in fact been transferred to the boundary, giving rise to Problem (1.3) involving the homogeneous elliptic equation subject to a new boundary condition. The solution of Problem (1.3) can then be found by standard boundary techniques [15–17]. The solution u of Problem (1.1) is then obtained as $u = u_p + u_h$.

Following this approach we face the challenge of approximating f in such a way that also allows us to find a particular solution. For general differential operators, the method of particular solution (MPS) [17] has been used to overcome the difficulties of evaluating u_p. It allows for the decoupling of the original given problem (1.1) into a particular solution and a homogeneous solution. In the framework of the MPS, a variety of basis functions can be used to approximate the source function. Most commonly, the source function is approximated by a series of radial basis functions (RBFs). For example, Partridge et al. in [25] and Muleshkov et al. in [22] used the RBF approximation for the Laplacian and Helmholtz-type operators, respectively. Despite the important interpolating properties of RBFs, one of their drawbacks is that it is difficult to obtain rapidly convergent RBF interpolants. As a consequence, one often has to use a large number of interpolating points, which could lead to a large, dense and highly ill-conditioned system of equations.

Other classes of approximations, such as trigonometric [1] and polynomial [6, 13, 18] ones, have been considered to overcome the difficulties encountered in the use of RBFs in the MPS. Chen et al in [6] obtained particular solutions in analytical form for the 2-D Poisson equation when the source function f is a homogeneous polynomial. Golberg et al. in [18] implemented the MPS when they used Chebyshev interpolants in their approach. In [18], particular solutions in analytical form for 2-D and 3-D Helmholtz-type equations when the source function is a monomial and for the 3-D Poisson equation when the source function is a homogeneous polynomial are obtained. Symbolic software packages such as Maple and Mathematica can be used for the implementation of the algorithm to get particular solutions. However, after the source function f is approximated