Commun. Comput. Phys.
doi: 10.4208/cicp.050610.021210a

Vol. **10**, No. 5, pp. 1161-1183
November 2011

Split Local Artificial Boundary Conditions for the Two-Dimensional Sine-Gordon Equation on $\mathbb{R}^2$

Houde Han* and Zhiwen Zhang

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China.

Received 5 June 2010; Accepted (in revised version) 2 December 2010

Available online 2 August 2011

Abstract. In this paper the numerical solution of the two-dimensional sine-Gordon equation is studied. Split local artificial boundary conditions are obtained by the operator splitting method. Then the original problem is reduced to an initial boundary value problem on a bounded computational domain, which can be solved by the finite difference method. Several numerical examples are provided to demonstrate the effectiveness and accuracy of the proposed method, and some interesting propagation and collision behaviors of the solitary wave solutions are observed.

AMS subject classifications: 52B10, 65D18, 68U05, 68U07

Key words: Sine-Gordon equation, operator splitting method, artificial boundary condition, soliton, unbounded domain.

1 Introduction

The sine-Gordon equation is a nonlinear hyperbolic partial differential equation, which was originally considered in the nineteenth century in the course of study of surfaces of constant negative curvature. Then it attracted a lot of attention in the 1970s due to the presence of soliton solutions. In recent years the sine-Gordon equation has also been used to describe physical models which possess soliton-like structures in higher dimensions. A typical example is the Josephson junction model which consists of two layers of super conducting material separated by an isolating barrier [1, 2]. This paper is devoted to study the numerical solution of the two-dimensional sine-Gordon equation on $\mathbb{R}^2$.

The initial value problem of the two-dimensional sine-Gordon equation is given by the following problem:

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}+\sin(u)=0, \quad x,y\in\mathbb{R}^1, \quad t>0, \tag{1.1a}$$

*Corresponding author. *Email addresses:* hhan@math.tsinghua.edu.cn (H. Han), zhangzhiwen02@mails.tsinghua.edu.cn (Z. Zhang)

$$u|_{t=0}=\varphi_0(x,y),\quad u_t|_{t=0}=\varphi_1(x,y),\quad x,y\in\mathrm{R}^1, \tag{1.1b}$$

where $u=u(x,y,t)$ represents the wave displacement at position (x,y) and at time t, $\varphi_0(x,y)$, $\varphi_1(x,y)$ are the initial displacement and velocity respectively, and $\sin(u)$ is the nonlinear force term.

The essential difficulty of the numerical solution to the problem (1.1a)-(1.1b) involves two parts, the nonlinearity of the equation and the unboundedness of the physical domain. For the bounded domain case, there are a lot of studies on the numerical solution of the two-dimensional sine-Gordon equation with Dirichlet, Neumann or periodic boundary condition. For example, Guo et al. [5] used finite difference scheme to investigate the numerical solution of the sine-Gordon equation with periodic boundary condition. Xin [6] studied the sine-Gordon equation as an asymptotic reduction of the two level dissipationless Maxwell-Bloch system. In addition, Christiansen and Lomdahl [7] used a generalized leap frog method, Argyris et al. [8] used a finite element approach, Sheng et al. [9] adopted a split cosine scheme, Djidjeli et al. [10] used a two-step one-parameter leapfrog scheme, A. G. Bratsos [11] used a three time-level fourth-order explicit finite-difference scheme, to simulate the sine-Gordon equation. Recently, A. G. Bratsos [12] adopted the method of lines to solve the two-dimensional sine-Gordon equation. M. Dehghan and A. Shokri [13] used the radial basis functions to solve the two-dimensional sine-Gordon equation. D. Mirzaei and M. Dehghan [14] used the continuous linear elements to obtain the boundary element solution of the two-dimensional sine-Gordon equation. However, when one wants to solve the two-dimensional sine-Gordon equation on the unbounded domain, these methods will face essential difficulties. Since the unboundedness of the physical domain in the problem (1.1a)-(1.1b), the standard finite element method or finite difference method cannot be used directly. In this paper, we will consider the numerical solution of the two-dimensional sine-Gordon equation on the unbounded domain.

The artificial boundary condition (ABC) method is a powerful approach to reduce the problems on the unbounded domain to a bounded computational domain. In general, the artificial boundary conditions can be classified into implicit boundary conditions and explicit boundary conditions including global, also called nonlocal ABC, local ABC and discrete ABC [4]. For the last thirty years, many mathematicians have made great contributions on this subject, see [15–19], which makes the artificial boundary condition method for the linear partial differential equations on the unbounded domain become a well developed method. In recent few years, there have been some new progress on the artificial boundary condition method for nonlinear partial differential equations on the unbounded domain. H. Han et al. [20] and Z. Xu et al. [21] used the Cole-Hopf transformation to get the exact ABCs for the viscous Burger's equation and the deterministic KPZ equation. Zheng [23, 24] adopted the inverse scattering approach to obtain the exact ABCs for the one-dimensional cubic nonlinear Schrödinger equation and the one-dimensional sine-Gordon equation. H. Han et al. [22] also utilized an operator splitting method to design split local artificial boundary conditions for the one-dimensional