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WENO Schemes for Mixed-Element Unstructured Meshes

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Abstract. The paper extends weighted essentially non-oscillatory (WENO) schemes to two-dimensional quadrilateral and mixed-element unstructured meshes. The key element of the proposed methods is a reconstruction procedure suitable for arbitrarily-shaped cells. The resulting schemes achieve the designed uniformly high-order of accuracy and compute discontinuous solutions without spurious oscillations at interfaces between cells of two different types.

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1 Introduction

Weighted essentially non-oscillatory (WENO) schemes [1,6,10,11,14,19] represent a popular class of high-order methods for hyperbolic conservation laws. These methods combine very high-order of spatial accuracy when the solution is smooth and at the same time do not produce spurious oscillations near sharp gradients or across discontinuities. Originally, the schemes were developed and applied on structured Cartesian meshes. Unstructured meshes are more suitable for practical applications with complicated geometries. Therefore, WENO schemes have been extended to two-dimensional triangular meshes [6, 10] and three-dimensional tetrahedral meshes [3, 4, 25]. Accurate results were obtained for a number of difficult test cases thus showing the potential of the methods.

The aim of the present work is to develop finite-volume WENO methods both for two-dimensional quadrilateral and mixed-element unstructured meshes, comprising quadrilateral and triangular elements. The motivation is twofold. Firstly, numerical modelling of viscous flows often requires the use of mixed-element meshes in which quadrilateral

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cells are typically used inside the boundary layers where the rest of the computational domain is discretised by triangular cells. Therefore, numerical methods should be able to use unstructured meshes consisting of elements of different types and not purely triangular ones. Secondly, the use of quadrilateral meshes may lead to better computational efficiency due to the use of fewer elements and larger time step.

The paper is organised as follows. In Section 2 the general framework of finite-volume WENO schemes on mixed-element meshes is outlined. In Section 3 a detailed explanation of both linear and nonlinear reconstruction for a scalar variable is provided. This reconstruction can be viewed as an extension of the WENO methodology proposed in [3, 4]. The application of the developed technique to the compressible Euler equations is discussed in Section 4. Section 5 presents numerical results, which demonstrate the very high-order accuracy of the resulting methods, their essentially non-oscillatory properties as well as illustrate the influence of the mesh quality on the accuracy of the calculations. Conclusions are drawn in Section 6.

2 The numerical framework

Consider a two-dimensional hyperbolic system of the form

$$\frac{\partial}{\partial t}\mathbf{U}+\frac{\partial}{\partial x}\mathbf{F}(\mathbf{U})+\frac{\partial}{\partial y}\mathbf{G}(\mathbf{U})=\mathbf{0}, \tag{2.1}$$

where $\mathbf{U}$ is the vector of conserved variables; $\mathbf{F}$, $\mathbf{G}$ are flux vectors in the $x-$ and $y-$ coordinate directions, respectively. Suppose that the computational domain is discretised by conforming elements E_i of the area $|E_i|$, indexed by a unique mono-index i. Vertices $\mathbf{x}^{(l)}=(x^{(l)},y^{(l)})$ and sides of each element are indexed by $l=1,\cdots,L(i)$ with $\mathbf{x}^{(L(i)+1)}=\mathbf{x}^{(1)}$. The centre of the element has coordinates (x_i,y_i).

Integrating (2.1) over an element i in space, the following semi-discrete finite-volume method is obtained :

$$\frac{d}{dt}\bar{\mathbf{U}}_i+\frac{1}{|E_i|}\int\limits_{\partial E_i}\mathbf{H}_n ds=\mathbf{0}, \quad \mathbf{H}_n=n_x\mathbf{F}+n_y\mathbf{G}, \tag{2.2}$$

where $\mathbf{n}=(n_x,n_y)$ is the outward unit normal vector of the cell E_i and $\bar{\mathbf{U}}_i(t)$ are the cell averages of the solution at time t. The integral over the boundary ∂E_i is split into the sum of integrals over each side l resulting in the following expression:

$$\frac{d}{dt}\bar{\mathbf{U}}_i=\mathbf{R}_i, \quad \mathbf{R}_i=-\frac{1}{|E_i|}\sum_{l=0}^{L(i)}\int\limits_{\mathbf{x}^{(l)}}^{\mathbf{x}^{(l+1)}}\mathbf{H}_n ds=-\frac{1}{|E_i|}\sum_{l=0}^{L(i)}\mathbf{H}_{il}. \tag{2.3}$$

In the computational framework, the intercell flux $\mathbf{H}_{il}$ corresponding to the side l of the cell E_i is approximated by a suitable Gaussian numerical quadrature:

$$\mathbf{H}_{il}=\sum_{\beta}\mathbf{H}_n\left(\mathbf{U}(\mathbf{x}_\beta,t)\right)\omega_\beta|A_l|, \tag{2.4}$$