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Derivation of a Non-Local Model for Diffusion Asymptotics — Application to Radiative Transfer Problems

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Abstract. In this paper, we introduce a moment closure which is intended to provide a macroscopic approximation of the evolution of a particle distribution function, solution of a kinetic equation. This closure is of non local type in the sense that it involves convolution or pseudo-differential operators. We show it is consistent with the diffusion limit and we propose numerical approximations to treat the non local terms. We illustrate how this approach can be incorporated in complex models involving a coupling with hydrodynamic equations, by treating examples arising in radiative transfer. We pay a specific attention to the conservation of the total energy by the numerical scheme.

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1 Introduction

We are interested in the derivation of "intermediate models", intended to capture the essential features of the solutions of the following kinetic equation

$$\varepsilon\partial_t f_\varepsilon + a(v)\cdot\nabla_x f_\varepsilon = \frac{1}{\varepsilon}Q(f_\varepsilon), \tag{1.1}$$

for small values of the parameter $\varepsilon>0$. This equation arises when describing the evolution of many "particles" described by their distribution function in phase space $f_\varepsilon(t,x,v):\ t\geq$

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0, $x \in \mathbb{R}^N$, stand for the time and space variables, respectively while the variable v — which lies in some measured space $(\mathcal{V}, \mathrm{d}v)$, $\mathcal{V} \subset \mathbb{R}^M$ — represents some physical state of the particles. The word "particles" might refer to very different physical situations: they could be gas molecules, neutrons, electrons or ions, photons, stars or planets or even bacteria etc. In this paper, we shall be mainly concerned with the case where (1.1) models the evolution of radiation energy (but our approach applies to a general framework). The function $a: \mathcal{V} \to \mathbb{R}^N$ associates to a state v the velocity $a(v)$ and the left hand side in (1.1) describes the transport of the particles whereas Q is a "collision operator" intended to describe the interaction processes the particles are subject to. In most of the applications this operator acts only on the variable v. The most common framework is

$$a(v) = v \in \mathbb{R}^N, \quad \mathcal{V} = \mathbb{R}^N, \quad \text{or} \quad \mathcal{V} = \mathbb{S}^{N-1},$$

endowed with the Lebesgue measure, but more complicated models can be dealt with. The Eq. (1.1) is written in dimensionless form and the parameter ε is associated to the mean free path of the particles. As $\varepsilon \to 0$ the solution of (1.1) tends towards an equilibrium state which makes the collision operator vanish

$$f_\varepsilon \simeq f_{\mathrm{eq}}, \quad Q(f_{\mathrm{eq}}) = 0.$$

Therefore, this fixes the dependence with respect to the variable v, and the dynamics reduces to a diffusion equation satisfied by some macroscopic quantities, which means some v-average of the unknown.

However for many applications, such a convergence statement is not sufficient; instead, one would be interested in a model for intermediate regimes, for small, but non zero ε's. For instance, the diffusion equation propagates information with infinite speed, while the speed of propagation of the original model is v/ε. There exists a huge literature on possible ways to derive a model "in-between" the fully microscopic description and the limit diffusion equation. The question is particularly relevant for numerical purposes since computing the solution of (1.1) for small ε's becomes rapidly non affordable. Very often these reduced models are constructed with a suitable closure of the equations satisfied by moments with respect to v of f_ε. We refer for instance to [32, 33, 36] for presentation and comparison of such closures. Among others the closure based on entropy minimization principle, which is referred to as the $M1$ model, received a lot of attention, see [20, 21, 26, 34, 35]; it is used for numerical simulations in radiative transfer [8–10]. The well-posedness and consistency with the diffusion asymptotics have been established rigorously in [15] and for numerical experiments we refer to [13]. For recent progress on moment closures, we refer also to [53].

This work is devoted to another class of reduced models which are of nonlocal or, more precisely, of pseudo-differential nature. These nonlocal models have been introduced for complex kinetic equations arising in plasma physics, and they are currently used in numerical simulations of fusion plasmas. The introduction of such models dates back to [42, 43], and we refer to further discussions and improvements to [1, 24, 25, 45, 47, 50, 51, 57]. The objectives of the present contribution can be summarized as follows: