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Accuracy Enhancement Using Spectral Postprocessing for Differential Equations and Integral Equations

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Abstract. It is demonstrated that spectral methods can be used to improve the accuracy of numerical solutions obtained by some lower order methods. More precisely, we can use spectral methods to postprocess numerical solutions of initial value differential equations. After a few number of iterations (say 3 to 4), the errors can decrease to a few orders of magnitude less. The iteration uses the Gauss-Seidel type strategy, which gives an explicit way of postprocessing. Numerical examples for ODEs, Hamiltonian system and integral equations are provided. They all indicate that the spectral processing technique can be a very useful way in improving the accuracy of the numerical solutions. In particular, for a Hamiltonian system accuracy is only one of the issues; some other conservative properties are even more important for large time simulations. The spectral postprocessing with the coarse-mesh symplectic initial guess can not only produce high accurate approximations but can also save a significant amount of computational time over the standard symplectic schemes.

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Key words: Postprocessing, spectral methods, rate of convergence, Hamiltonian system, integral equations.

1 Introduction

We begin by considering a simple ordinary differential equation with given initial value:

$$y'(x)=g(y;x), \quad 0<x\le T, \tag{1.1}$$

$$y(0)=y_0. \tag{1.2}$$

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There have been many numerical methods for solving (1.1)-(1.2), see, e.g., [9,10]. However, most of the existing methods have an algebraic rate of convergence, i.e., $\mathcal{O}(h^\alpha)$, with $\alpha=1$ for the Euler method, and $\alpha=4$ for the RK4 method.

A natural question is can we obtain exponential (spectral) rate of convergence for solving problem (1.1)-(1.2)? For boundary value problems, the answer is positive and well known, see, e.g., [2,4,14]. However, for the initial value problem (1.1)-(1.2), spectral methods are not attractive due to the following reasons: The problem (1.1)-(1.2) is a local problem, so a global method (such as spectral method) will require larger storage (need to store all data in a fixed interval) and computational time (need to solve a linear system or a *nonlinear* system in case that F in (1.1) is nonlinear). These disadvantages makes the use of the spectral approach for problem (1.1)-(1.2) less attractive.

The motivation of this article is to propose a *spectral postprocessing technique* which uses the numerical solutions of a lower order method to serve as starting value of the spectral methods. Then we take a few Gauss-Seidel type iterations for a well designed spectral method. This postprocessing procedure will help us to recover the exponential rate of convergence with little extra computational resource. In particular, there is no need of solving a linear system or a nonlinear system in case that F in (1.1) is nonlinear. Moreover, the method is found extremely stable for the initial value problem (1.1)-(1.2).

2 Spectral postprocessing for initial value ODEs

2.1 Spectral postprocessing for an ODE equation

Assume the size of $[t_0,T]$ is not too big; otherwise a trick in Section 2.2 will be used. In this case, we introduce the linear coordinate transformation

$$x=\frac{T-t_0}{2}s+\frac{T+t_0}{2},\quad -1\le s\le 1, \tag{2.1}$$

and the transformations

$$Y(s)=y\left(\frac{T-t_0}{2}s+\frac{T+t_0}{2}\right),\quad G(Y;s)=g\left(Y;\frac{T-t_0}{2}s+\frac{T+t_0}{2}\right). \tag{2.2}$$

Then problem (1.1)-(1.2) becomes

$$Y'(s)=G(Y;s),\quad -1<s\le 1; \tag{2.3}$$

$$Y(-1)=y_0. \tag{2.4}$$

2.1.1 Chebyshev collocation approach

Let $\{s_j\}_{j=0}^N$ be the Chebyshev-Gauss-Lobatto points: $s_j=\cos(\pi j/N), 0\le j\le N$. We project G to the polynomial space $\mathcal{P}_N$:

$$G(Y;s)=\sum_{j=0}^{N}G(Y_j;s_j)F_j(s), \tag{2.5}$$