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Numerical Calculation of Monotonicity Properties of the Blow-Up Time of NLS

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Abstract. We investigate blow-up of the focusing nonlinear Schrödinger equation, in the critical and supercritical cases. Numerical simulations are performed to examine the dependence of the time at which blow-up occurs on properties of the data or the equation. Three cases are considered: dependence on the scale of the nonlinearity when the initial data are fixed; dependence upon the strength of a quadratic oscillation in the initial data when the equation and the initial profile are fixed; and dependence upon a damping factor when the initial data are fixed. In most of these situations, monotonicity in the evolution of the blow-up time does not occur.

AMS subject classifications: 35Q55, 65M70, 81Q05

Key words: Nonlinear Schrödinger equation, finite time blow-up, wave collapse.

1 Introduction

Finite time blow-up of Nonlinear Schrödinger equations (NLS) is an interesting topic that has drawn much attention. A (by far) non-exhaustive list of important papers on the topic is [10, 15–19, 23], and we refer to [21] for a nice survey of the latest results. As recalled in [21], the main three directions of research in this subject are: giving sufficient conditions to have finite time blow-up in the energy space; estimating the blow-up rate and the stability of the blow-up regimes; describing the spatial structure of the singularity formation.

The question addressed in this paper is to investigate the blow-up time and its possible relations to features of the equation or the data. This point of view differs from much of the other work on the subject of blow-up in the sense that it is not concerned only with properties of the solution "close" to the blow-up, but also needs to take into account parts

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of the solution which actually are "far away" (either in space or in time) from the actual blow-up region.

The present paper is an extension of [8], where a detailed study of the relevant analytic results is done in addition to the numerical tests.

Consider the nonlinear Schrödinger equation

$$i\partial_t u+\Delta u=-\lambda|u|^{2\sigma}u \ , \ (t,x)\in\mathbb{R}_+\times\mathbb{R}^n; \quad u_{|t=0}=u_0, \tag{1.1}$$

with a focusing power nonlinearity, where $\lambda,\sigma>0$. Such an equation arises in nonlinear optics as an envelope equation in the propagation of laser beams (see, e.g., [22]), and also in applications of quantum mechanics, where other terms like confining potentials and coupling to Poisson's equations are usually also a part of the model equation. It is well known that if $\sigma<\frac{2}{n-2}$, then for $u_0\in H^1(\mathbb{R}^n)$ the equation (1.1) has a unique solution in $H^1(\mathbb{R}^n)$, defined locally in time. In general this solution does not remain in $H^1(\mathbb{R}^n)$ globally in time, when $\sigma\geq\frac{2}{n}=\sigma_{crit}$, finite time blow-up may occur ($\lambda>0$ means that the nonlinearity is focusing). For proofs of these standard results we refer to the monographs [11, 22]. The L^2-norm or mass of $u(t,\cdot)$ is independent of time, and finite time blow-up means that there exists $T^*<\infty$ such that:

$$\|\nabla_x u(t)\|_{L^2}\to+\infty \quad \text{as } t\to T^*.$$

In this paper, we investigate by numerical experiments the dependence of the blow-up time upon, for instance, a varying coupling constant λ when the initial datum u_0 is fixed. To motivate our study, we recall now some results from [12, 13]. In [12], the authors prove that if the initial datum $u_0(x)$ is replaced by $u_0(x)e^{-ib|x|^2/4}$, then the blow-up time of the corresponding new solution u_b can be related explicitly to that of u, in the case of a critical nonlinearity, $\sigma=\frac{2}{n}$. It is a consequence of the conformal invariance. In the super-critical case $\sigma>\frac{2}{n}$, the conformal transform does not leave (1.1) invariant. It is also established that if u has negative energy (in this case, there is finite time blow-up at least if $xu_0\in L^2(\mathbb{R}^n)$ [14]) then *for large* b, blow-up occurs sooner than for $b=0$; unlike in the conformally invariant case, one does not know whether the blow-up time is monotonous with respect to b. The numerical experiments we present here show that it is not monotonous with respect to b.

In [13], the author considers the damped cubic Schrödinger equation in space dimension two:

$$i\partial_t\psi+\Delta\psi=-i\delta\psi-|\psi|^{2\sigma}\psi \ , \ (t,x)\in\mathbb{R}_+\times\mathbb{R}^2; \quad \psi_{|t=0}=u_0, \tag{1.2}$$

with $\sigma=1$. It is conjectured that the blow-up time is monotonous with respect to $\delta>0$. Our numerical experiments show that this guess is not satisfied. The guess is plausible when one thinks of the initial data u_0 as a single hump, for example a gaussian. In this case the experiments show monotonicity; however when the data u_0 is made of, say, two humps, the intuition goes wrong.

In addition to the linear damping term in (1.2), often also a cubic or quintic nonlinear damping term is considered, for example in BEC modeling. A study of this case is beyond