COMMUNICATIONS IN COMPUTATIONAL PHYSICS
Vol. 6, No. 4, pp. 883-902

Commun. Comput. Phys.
October 2009

Parareal-Richardson Algorithm for Solving Nonlinear ODEs and PDEs

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Received 28 September 2008; Accepted (in revised version) 11 March 2009

Available online 6 April 2009

Abstract. The parareal algorithm, proposed firstly by Lions *et al.* [J. L. Lions, Y. Maday, and G. Turinici, A "parareal" in time discretization of PDE's, C.R. Acad. Sci. Paris Sér. I Math., 332 (2001), pp. 661-668], is an effective algorithm to solve the time-dependent problems parallel in time. This algorithm has received much interest from many researchers in the past years. We present in this paper a new variant of the parareal algorithm, which is derived by combining the original parareal algorithm and the Richardson extrapolation, for the numerical solution of the nonlinear ODEs and PDEs. Several nonlinear problems are tested to show the advantage of the new algorithm. The accuracy of the obtained numerical solution is compared with that of its original version (i.e., the parareal algorithm based on the same numerical method).

AMS subject classifications: 65Y05, 65Y10, 65Y20, 37M05, 65-05

Key words: Parallel computation, parareal algorithm, Richardson extrapolation, accuracy, nonlinear problems.

1 Introduction

In the seminal paper [24] the concept of a new domain decomposition for the numerical solution of time-dependent problems, the parareal algorithm, was proposed by Lions, Maday and Turinici. The name, *parareal*, was chosen to indicate that the algorithm is constructed to compute simultaneously in time the solution of evolution problems whose solution cannot be obtained in real time using one processor only. The method has received much interest from many researchers in the past years, especially in the area of domain decompositions, see, e.g., [21]. Many excellent results about this algorithm have been obtained and below we will make a brief retrospection.

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The idea targeted a large scale time parallelism of simulating the evolution problems was proposed already in 1964 by Nievergelt [29]. The idea of Nievergelt eventually developed into the well known multiple shooting method for boundary value problems [20]. For much more investigation in this direction we refer the interested reader to [4, 5, 22]. Later in 1967, Miranker and Liniger [26] introduced a family of naturally Runge-Kutta methods for small scale parallelism coupled with predictor-corrector strategies. The parallelization of these methods lays on that, the prediction and correction steps can be performed simultaneously over several time steps.

The parareal algorithm was first introduced in [24] and an improved version was given by Bal and Maday in [2]. Some further improvements and understanding, as well as new applications of this algorithm, were investigated by Baffico *et al.* in [3] and Maday and Turinici in [27,28]. Its stability was investigated in [1,30]. Recently, several variants of this algorithm have been proposed in [6, 10, 14] and extensive experiments can be found for fluid and structure problems in [6, 10], for the Navier-Stokes equations in [7, 8], for reservoir simulation in [11], and for various nonlinear problems, such as Brusselator, Arenstorf orbit and viscous Burgers' equation etc. in [13].

We pay special attention to the recent results presented by Gander and Vandewalle in [12]. In that paper, the relation of the parareal algorithm to the space-time multigrid methods [16,18,19,25,31–33] and multiple shooting methods was first investigated. It has been shown that the parareal algorithm can be regarded as the practical implementation of the multiple shooting and time-multigrid methods. The new convergence results that show superlinear convergence of the algorithm on bounded time intervals and linear convergence on unbounded intervals were also presented in that paper. It also provides a up-to-date historical review and references in this field.

Along the lines of [12], we investigate in this paper a new variant of the parareal algorithm, namely Parareal-Richardson algorithm, for the time dependent problems. The new algorithm is derived by combining the original parareal algorithm and the Richardson extrapolation, and hence the accuracy of the numerical solution obtained by the proposed algorithm is higher than that of the original parareal algorithm. The aim of this paper is to show the advantages of this new algorithm in terms of the accuracy when applied to nonlinear ODEs and PDEs. Moreover, the advantages with respect to the stability and convergence rate for the proposed algorithm are presented.

The remainder of this paper is organized as follows. The Parareal-Richardson algorithm is described in detail in Section 2. We also discuss the stability property and the accuracy of the Parareal-Richardson scheme. It is demonstrated that for some one-step numerical methods, the stability region of the Parareal-Richardson algorithm is larger than that of the parareal algorithm. In Section 3, we apply the parareal and Parareal-Richardson algorithms to several classical nonlinear ODEs and PDEs, and to demonstrate that the Parareal-Richardson algorithm is more flexible and outperforms the parareal algorithm. In Section 4, we discuss the effects of the parameters used in the proposed algorithm to the convergence speed. In Section 5, we give some conclusions of this paper and discuss future directions of the research for the Parareal-Richardson algorithm.