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An Edge-Based Anisotropic Mesh Refinement Algorithm and its Application to Interface Problems

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Abstract. Based on an error estimate in terms of element edge vectors on arbitrary unstructured simplex meshes, we propose a new edge-based anisotropic mesh refinement algorithm. As the mesh adaptation indicator, the error estimate involves only the gradient of error rather than higher order derivatives. The preferred refinement edge is chosen to reduce the maximal term in the error estimate. The algorithm is implemented in both two- and three-dimensional cases, and applied to the singular function interpolation and the elliptic interface problem. The numerical results demonstrate that the convergence order obtained by using the proposed anisotropic mesh refinement algorithm can be higher than that given by the isotropic one.

AMS subject classifications: 65N22, 65N50, 65N55

Key words: Adaptive finite element method, anisotropic mesh refinement, elliptic interface problem, non-homogeneous jump, *a posteriori* error estimate.

1 Introduction

For singular or nearly singular problems, the structures of singularity often exhibit ”low-dimensional” feature that the solutions vary significantly in some directions but mildly in other directions. To numerically approximate such solutions efficiently, no doubt we prefer anisotropic meshes, which are of different length scales in different directions and fit the anisotropic feature in the solutions. Numerous examples, including

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pervasive layer structures and interface discontinuities, have shown the efficiency of anisotropic elements in reducing computational cost and improving approximation accuracy [1, 5, 6, 17, 22, 27, 28, 30]. This paper is concerned with the elliptic interface problem with homogeneous and non-homogeneous jump conditions, which attracts much interests since it is omnipresent in many scientific and engineering problems, including multi-phase flows, nano-electronic devices, electromagnetic wave propagation in heterogeneous waves, implicit solvent models in structural biology, and biological membrane. To resolve the layer anisotropy, we develop an anisotropic refinement algorithm which can be effective not only for the interface problem but also for problems with global anisotropy.

Compared with isotropic elements, the description of anisotropic meshes needs more information. Take two-dimensional triangular element as example, its anisotropy can be measured in two main aspects [8]. One is orientation, which is roughly the direction of its longest side. The other is the aspect ratio, which measures how thin the triangle is. The first quantity is supposed to be more crucial to the success of anisotropic element. Goodman et al. [15] once gave an example showing that a wrong direction may lead to non-convergence. In the past decades, some important improvement has been made in numerical analysis of linear interpolation on anisotropic triangular meshes [3, 7, 19, 23]. The main conclusion can be roughly stated as: given the area of a triangular element τ, the error (in L^p-norm) for the linear interpolation of a function u at the vertices of τ is nearly the minimum when τ is aligned with the eigenvector (associated with the smaller eigenvalue) of the Hessian $\nabla^2 u$, and the aspect ratio (or stretch ratio) of τ is about the square root of the ratio of the greater eigenvalue of $\nabla^2 u$ to the smaller one. For quadratic interpolation, the anisotropic orientation depends on $\nabla^3 u$ [8] and higher order interpolation may have similar properties. Based on this analysis, some anisotropic mesh optimization methodologies have been developed [4, 9, 16], which try to minimize the error by relocating nodes. From a practical point of view, since the solution of the problem is unknown, a crucial point in these methods is how to approximate high order derivatives such as $\nabla^2 u$ or $\nabla^3 u$ efficiently and accurately. When the solution is not regular enough, the accuracy in recovering the high order derivatives can be misleading.

As an effort to overcome the difficulty of requiring high order derivatives, the method we propose depends on only the first order derivatives of $u(u_h)$. For each element with indicator above the given tolerance, one edge is chosen as the preferred refinement edge. The affine map from the reference element to the actual element plays an essential role in anisotropic error analysis. In [11, 12], Formaggia et al. proved that the sum of error gradient projection onto two principal axes of the affine map is an upper bound of the element error. We project the error gradient onto the element edges instead to find the preferred refinement edge. Since the Jacobian matrix of the affine map can be expressed by edge vectors when we use the unit reference triangle, the sum of error gradient projection onto the three edges is again an upper bound of the element error. To reduce this upper bound error estimate, the most efficient mesh adaptation is to refine the edge with the maximal contribution to the estimate. The algorithm is first validated for the inter-