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The Direct Discontinuous Galerkin (DDG) Method for Diffusion with Interface Corrections

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Abstract. Based on a novel numerical flux involving jumps of even order derivatives of the numerical solution, a direct discontinuous Galerkin (DDG) method for diffusion problems was introduced in [H. Liu and J. Yan, SIAM J. Numer. Anal. 47(1) (2009), 475-698]. In this work, we show that higher order ($k\geq 4$) derivatives in the numerical flux can be avoided if some interface corrections are included in the weak formulation of the DDG method; still the jump of $2nd$ order derivatives is shown to be important for the method to be efficient with a fixed penalty parameter for all p^k elements. The refined DDG method with such numerical fluxes enjoys the optimal $(k+1)$th order of accuracy. The developed method is also extended to solve convection diffusion problems in both one- and two-dimensional settings. A series of numerical tests are presented to demonstrate the high order accuracy of the method.

AMS subject classifications: 35K05, 35K15, 65N12, 65N30

Key words: Diffusion, discontinuous Galerkin methods, stability, numerical flux.

1 Introduction

This paper is the continuation of our project, initiated in [26], of developing a direct discontinuous Galerkin (DDG) method for diffusion problems. Here we focus on the diffusion equation of the form

$$\partial_t U - \nabla\cdot(A(U)\nabla U)=0, \quad \Omega\times(0,T), \tag{1.1}$$

where $\Omega\subset\mathbb{R}^d$, the matrix $A(U)=(a_{ij}(U))$ is symmetric and positive definite, and U is an unknown function of (x,t). The method will also be extended to convection-diffusion problems and their invariants.

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The Discontinuous Galerkin (DG) method is a finite element method using a completely discontinuous piecewise polynomial space for the numerical solution and the test functions. One main advantage of the DG method was the flexibility afforded by local approximation spaces combined with the suitable design of numerical fluxes crossing cell interfaces. The application to hyperbolic problems has been quite successful since it was originally introduced by Reed and Hill [28] in 1973 for neutron transport equations. A major development of the DG method for nonlinear hyperbolic conservation laws is carried out by Cockburn, Shu, and collaborators in a series of papers [13, 17, 18, 20]. We refer to [11, 16, 21] for reviews and further references.

However, the application of the DG method to diffusion problems has been a challenging task because of the subtle difficulty in defining appropriate numerical fluxes for diffusion terms, see e.g. [30]. There have been several DG methods suggested in literature to solve the problem, including the method originally proposed by Bassi and Rebay [4] for compressible Navier-Stokes equations, its generalization called the local discontinuous Galerkin (LDG) methods introduced in [19] by Cockburn and Shu and further studied in [6,7,12,15]; as well as the method introduced by Baumann-Oden [5,27]. Also in the 1970s, Galerkin methods for elliptic and parabolic problems using discontinuous finite elements, called the *interior penalty* (IP) methods, were independently introduced and studied; see, e.g., [1,3,34]. We refer to [2] for a unified analysis of DG methods for elliptic problems and background references for the IP methods.

In this article we are interested in the effect of test functions on interface treatments, and accordingly we introduce a refined version of the DDG method proposed in [26]. To illustrate the idea, we consider the scalar one-dimensional diffusion equation

$$u_t = u_{xx},$$

and formulate the DDG method based on the direct weak formulation

$$\int_{I_j} u_t v dx - \widehat{(u_x)} v \Big|_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} + \int_{I_j} u_x v_x dx = 0,$$

where I_j is the j-th computational cell, and v is the test function. In [26] we presented the following numerical flux

$$\widehat{u_x} = \beta_0 \frac{[u]}{\Delta x} + \overline{u_x} + \beta_1 \Delta x [u_{xx}] + \beta_2 (\Delta x)^3 [u_{xxxx}] + \cdots, \tag{1.2}$$

which involves the average $\overline{u_x}$ and the jumps of even order derivatives of u. This numerical flux satisfies the following desired properties: it (i) is consistent for the smooth solution u; (ii) is conservative in the sense of its being single valued at the interface; (iii) ensures the L^2-stability; and (iv) enforces the high order accuracy of the method

It was shown in [26] that for piecewise p^k polynomial approximations, kth order of accuracy of the DDG method is ensured if the numerical flux is admissible. Numerical experiments in [26] also showed that the use of term $(\Delta x)^{2m-1}[\partial_x^{2m} u]$ $(m = 0,1,\cdots,[\frac{k}{2}])$ in