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Quasi-Lagrangian Acceleration of Eulerian Methods

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Abstract. We present a simple and efficient strategy for the acceleration of explicit Eulerian methods for multidimensional hyperbolic systems of conservation laws. The strategy is based on the Galilean invariance of dynamic equations and optimization of the reference frame, in which the equations are numerically solved. The optimal reference frame moves (locally in time) with the average characteristic speed of the system, and, in this sense, the resulting method is quasi-Lagrangian. This leads to the acceleration of the numerical computations thanks to the optimal CFL condition and automatic adjustment of the computational domain to the evolving part of the solution. We show that our quasi-Lagrangian acceleration procedure may also reduce the numerical dissipation of the underlying Eulerian method. This leads to a significantly enhanced resolution, especially in the supersonic case. We demonstrate a great potential of the proposed method on a number of numerical examples.

AMS subject classifications: 76M12, 65B99, 35L65

Key words: Eulerian methods, quasi-Lagrangian method, finite volume methods, Galilean invariance.

1 Introduction

We study numerical methods for hyperbolic systems of conservation laws, which, in the one-dimensional (1-D) case, read:

$$\mathbf{u}_t+\mathbf{f}(\mathbf{u})_x=\mathbf{0}, \tag{1.1}$$

where $\mathbf{u}(x,t):=(u^{(1)}(x,t),u^{(2)}(x,t),\cdots,u^{(N)}(x,t))^T$ is an N-dimensional vector of unknowns and $\mathbf{f}(\mathbf{u}(x,t)):=(f^{(1)}(\mathbf{u}(x,t)),f^{(2)}(\mathbf{u}(x,t)),\cdots,f^{(N)}(\mathbf{u}(x,t)))^T$ is the flux function. We restrict our consideration to initial value problems (IVP) and initial-boundary value problems with periodic boundary conditions.

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There are two big classes of the numerical methods for (1.1): Eulerian and Lagrangian ones. The main advantage of Eulerian methods is that they employ stationary spatial grids, which makes numerical flux approximation relatively easy. In the Lagrangian framework, the grid moves together with the medium, which typically leads to a very high resolution of contact waves, but makes the computation of numerical fluxes substantially more involved.

Another drawback of Lagrangian methods is a lack of control of the developed grid structure: the grid, which is moving with the fluid, may become highly irregular. This would affect both the efficiency of the method and its accuracy. One of the ways to overcome this difficulty is to use ALE methods (see, e.g., [1,16,19] and references therein), in which the computed solution is projected onto the regular grid after each time step or after every few time steps so that one makes sure that the mesh does not become highly irregular. Another way of fixing the problem of irregular grid formation while enjoying the main advantage of Lagrangian methods—automatic adaptivity of the grid to the structure of the computed solution—is to use the adaptive moving mesh methods (see, e.g., [6,7,15,22] and references therein). In these methods, the mesh is moving not with the fluid, but according to a moving mesh PDE, [6,7,22], or the estimated local errors in the computed solution, [15].

In this paper, we only study Eulerian methods and focus on two specific issues: their efficiency and numerical dissipation. It is well-known that the CFL condition, related to the spectral radius of the Jacobian $\partial\mathbf{f}/\partial\mathbf{u}$, is a fundamental stability restriction on the size of time steps in Eulerian methods. We propose a general strategy for reduction of the CFL number for any given Eulerian method. The main idea is to use the Galilean invariance of the system (1.1), which allows one to choose, at each time step, the reference frame with the least restrictive CFL condition. The entire mesh is then shifted to stay in the selected frame. Notice, however, that unlike the case of Lagrangian or moving mesh methods, the structure of the mesh does not change at all by the proposed mesh shift.

In a nutshell, the strategy works as follows. At each time step, we add a linear advection term $-\sigma\mathbf{u}_x$ to the left-hand side of (1.1) and solve the resulting system

$$\mathbf{u}_t+\mathbf{f}(\mathbf{u})_x-\sigma\mathbf{u}_x=\mathbf{0}, \tag{1.2}$$

where σ is a constant. Obviously, solutions of (1.2) are obtained from the corresponding solutions of (1.1) by the change of variables $x\to x-\sigma t$. However, the constant σ provides us with an additional degree of freedom, and a wise choice of σ may help to improve both efficiency and resolution, achieved by the numerical method applied to (1.2) instead of (1.1). Our approach can be viewed as quasi-Lagrangian since σ is chosen so that the reference frame moves at the average characteristic velocity, as quantified in Section 2. We note that the proposed method is not a moving mesh method, but rather a "moving framework" one. It resembles a more sophisticated hybrid Eulerian-Lagrangian method from [24]. However, unlike the method from [24], our approach retains the simplicity of Eulerian methods. We would also like to mention that adding the linear convection term $-\sigma\mathbf{u}_x$ as it is done in (1.2) resembles the artificial wind method from [21]. However,