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Computational Investigation of the Effects of Sample Geometry on the Superconducting-Normal Phase Boundary and Vortex-Antivortex States in Mesoscopic Superconductors

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Abstract. A computational study of superconducting states near the superconducting-normal phase boundary in mesoscopic finite cylinders is presented. The computational approach uses a finite element method to find numerical solutions of the linearized Ginzburg-Landau equation for samples with various sizes, aspect ratios, and cross-sectional shapes, i.e., squares, triangles, circles, pentagons, and four star shapes. The vector potential is determined using a finite element method with two penalty terms to enforce the gauge conditions that the vector potential is solenoidal and its normal component vanishes at the surface(s) of the sample. The eigenvalue problem for the linearized Ginzburg-Landau equations with homogeneous Neumann boundary conditions is solved and used to construct the superconducting-normal phase boundary for each sample. Vortex-antivortex (V-AV) configurations for each sample that accurately reflect the discrete symmetry of each sample boundary were found through the computational approach. These V-AV configurations are realized just within the phase boundary in the magnetic field-temperature phase diagram. Comparisons are made between the results obtained for the different sample shapes.

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1 Introduction

There are many important technological and fundamental questions concerning magnetic properties and superconducting states in superconductors of mesoscopic scale. Recent advances in microfabrication and experimental techniques have made possible extensive studies of such small systems. In this paper, we provide a computational study of some of these questions.

When the system is in the normal conducting state, the external magnetic field fully penetrates the system. At a temperature below a critical temperature T_c, as the applied magnetic field decreases, a superconducting state nucleates in the system. Knowing this critical level of the magnetic field, *i.e.*, the nucleation field, one can construct the normal/superconducting (N/S) phase boundary, if that transition is of second order.

As is well known, a bulk superconductor is called type-I when $\kappa < 1/\sqrt{2}$ and type-II for $\kappa > 1/\sqrt{2}$, where κ is the Ginzburg-Landau parameter. If $\kappa < 0.417$ [1,2], we have the pure (homogeneous) superconducting state (the Meissner state) below the thermodynamic critical field H_c and the normal conducting state above H_c. For $0.417 < \kappa < 1/\sqrt{2}$, there exists a critical field H_{c3} such that a surface superconductivity state exists for $H_c < H < H_{c3}$ while the bulk of the sample remains in the normal conducting state; for $H > H_{c3}$, the whole sample is in the normal conducting state [1,2].

For bulk type-II superconductors, we have a mixed state and the Abrikosov vortex lattice is energetically favorable in the range $H_{c1} < H < H_{c2}$ [2], where H_{c2} is called the upper critical field, and H_{c1}, the lower critical field. We have the Meissner state below H_{c1} and a surface superconductivity state for $H_{c2} < H < H_{c3}$.

For circular cylindrical samples, surface superconductivity nucleates in the form of a giant vortex (GV) [3,4]. The order parameter takes the form $\psi = f(r)e^{iL\varphi}$, where the winding number L is a good quantum number; L is analogous to the orbital angular momentum of a particle. Calculations on the superconducting state in mesoscopic, type-I, superconducting thin films by solving the non-linear Gingzburg-Laudau equations (GL) at $H < H_{c2}$ have, in most cases, found transitions between GV states of different circulation quantum numbers L, with some multi-vortex (MV) states occasionally appearing as thermodynamically stable states, which become metastable below a critical radius [5–8]. Note that in general, L of a sample is defined as the sum of the winding numbers of all vortices minus the sum of the winding numbers of all antivortices in the sample, where the winding number of a vortex (antivortex) is defined to be the number of multiples of 2π (-2π) that the phase of the superconducting "pair-wave-function" order parameter ψ changes along a counterclockwise closed curve surrounding the vortex (antivortex).

As the size of sample becomes smaller and smaller, the geometry and topology of the sample has a fundamental influence on the superconducting state, and boundary effects increasingly dominate the nucleation of superconductivity.

The fluxoid quantization requirement gives rise to an oscillatory depression in the critical temperature T_c in cylindrical shells as described by the classical experiment of Little and Parks [9,10]. One should also recall that in thin films, the parallel critical field