Commun. Comput. Phys.
doi: 10.4208/cicp.190809.141209a

Vol. **8**, No. 3, pp. 663-689
September 2010

Navier-Stokes Spectral Solver in a Finite Cylinder

F. Auteri*, M. Biava and L. Quartapelle

Politecnico di Milano, Dipartimento di Ingegneria Aerospaziale, Via La Masa 34, 20156 Milano, Italy.

Received 19 August 2009; Accepted (in revised version) 14 December 2009

Communicated by Jie Shen

Available online 20 April 2010

Abstract. A primitive variable spectral method for simulating incompressible viscous flows inside a finite cylinder is presented. One element of originality of the proposed method is that the radial discretization of the Fourier coefficients depends on the Fourier mode, its dimension decreasing with the increase of the azimuthal modal number. This principle was introduced independently by Matsushima and Marcus and by Verkley in polar coordinates and is adopted here for the first time to formulate a 3D cylindrical Galerkin projection method. A second element of originality is the use of a special basis of Jacobi polynomials introduced recently for the radial dependence in the solution of Dirichlet problems. In this basis the radial operators are represented by matrices of minimal sparsity — diagonal stiffness and tridiagonal mass — provided here in closed form for the first time, and lead to a Helmholtz operator characterized by a favorable condition number. Finally, a new method is presented for eliminating the singular behaviour of the solution originated by the rotation of the lid with respect to the cylindrical wall. Thanks to these elements, the resulting Navier-Stokes spectral solver guarantees the differentiability to any order of the solution in the entire computational domain and does not suffer from the time-step stability restriction occurring in spectral methods with a point clustering close to the axis. Several test examples are offered that demonstrate the spectral accuracy of the solution method under different representative conditions.

AMS subject classifications: 65N30, 65N35

Key words: Navier-Stokes equations, finite cylindrical domain, spectral methods, Jacobi and Legendre polynomials, primitive variables, projection method.

---

1 Introduction

To simulate the flow inside cylindrical cavities or along straight tubes of circular cross section the use of cylindrical coordinates is the natural choice and the adoption of a spec-

*Corresponding author. *Email address:* auteri@aero.polimi.it (F. Auteri)

tral discretization is particularly convenient when high accuracy is a major concern. In the last two decades several numerical schemes of this type have been developed for solving the incompressible Navier-Stokes equations to study the stability and investigate the transition of flows within cylindrical walls.

Focusing directly on fully 3D, i.e., not axisymmetric, flows, the first successful spectral methods for solving the equations in cylindrical coordinates were introduced by Moser et al. [1] and by Marcus [2] to simulate the flow between concentric cylinders of infinite axial extent. For the more challenging situation of a finite cylindrical gap the first spectral scheme was developed by Le Quéré and Pécheux to reproduce natural convection flows by means of Chebyshev polynomials and using the influence matrix technique [3].

Coming to cylindrical domains including the axis, a first attempt for the axially periodic case was done by Quartapelle and Verri who proposed an uncoupled Chebyshev method employing integral conditions for pressure [4]. Then, for the very important case of a finite cylinder, the spectral method of Lopez et al. must be mentioned [5]. It is a projection method representing the full 3D extension of the axisymmetric spectral solver developed by Lopez and Shen [6]. These methods stem from a Galerkin formulation of the underlying elliptic equations and employ the hierarchical bases of Legendre polynomials leading to matrices of very small bandwidth introduced by Shen in [7].

When the computational domain includes the axis or part of it, any spatial discretization is faced with the difficulty that the system of cylindrical variables entails a coordinate singularity at the axis, the so-called "pole" or "centre problem". In fact, there are regularity conditions on the Fourier expansion coefficients to be respected on the axis to guarantee the infinite differentiability of scalar and vector functions there, as clarified by the analysis of Lewis and Bellan [8]. For spectral methods their fulfillment can ensure the spectral accuracy of the computed solutions.

Methods have been proposed in the last years for dealing with the axis problem in spectral approximations for incompressible viscous flows inside cylindrical walls. Just to mention two examples, Fornberg introduced a method consisting in extending the radial variable also to negative values [9], see also [10, Sec. 6.2, p. 110] or [11, Chap. 11, p. 115], Speetjiens and Clercx described a Chebyshev collocation method for the vorticity-velocity equations resorting to the influence matrix technique [12].

From the mathematical viewpoint of solving elliptic equations, the occurrence of the singularity on the axis in cylindrical coordinates was addressed originally by Mercier and Raugel for a finite-element-based approximation [14]. In the context of spectral methods, the issue has been considered in the monograph [15, Sec. 3.4.1, p. 90] and a variational formulation of scalar elliptic equations based on weighted Sobolev spaces is described in the monograph of Bernardi et al. [16].

As a matter of fact, the difficulties associated with the pole can be actually turned into an opportunity when discretizing the problem by means of a spectral approximation. According to the analysis of [8] the Fourier components $u^m(r,z)$ of a differentiable scalar function $u(r,z,\phi)$ of the cylindrical coordinates (r,z,ϕ) must satisfy the following