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Existence and Uniqueness of the Weak Solution of the Space-Time Fractional Diffusion Equation and a Spectral Method Approximation

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Abstract. In this paper, we investigate initial boundary value problems of the space-time fractional diffusion equation and its numerical solutions. Two definitions, i.e., Riemann-Liouville definition and Caputo one, of the fractional derivative are considered in parallel. In both cases, we establish the well-posedness of the weak solution. Moveover, based on the proposed weak formulation, we construct an efficient spectral method for numerical approximations of the weak solution. The main contribution of this work are threefold: First, a theoretical framework for the variational solutions of the space-time fractional diffusion equation is developed. We find suitable functional spaces and norms in which the space-time fractional diffusion problem can be formulated into an elliptic weak problem, and the existence and uniqueness of the weak solution are then proved by using existing theory for elliptic problems. Secondly, we show that in the case of Riemann-Liouville definition, the well-posedness of the space-time fractional diffusion equation does not require any initial conditions. This contrasts with the case of Caputo definition, in which the initial condition has to be integrated into the weak formulation in order to establish the well-posedness. Finally, thanks to the weak formulation, we are able to construct an efficient numerical method for solving the space-time fractional diffusion problem.

AMS subject classifications: 35S10, 35A05, 65M70, 65M12

Key words: Space-time fractional diffusion equation, existence and uniqueness, spectral methods, error estimates.

1 Introduction

Fractional partial differential equations (FPDEs) appear in the investigation of transport dynamics in complex systems which are characterized by the anomalous diffu-

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sion and nonexponential relaxation patterns [31]. Related equations of importance are the space/time fractional diffusion equations, the fractional advection-diffusion equation [17, 18] for anomalous diffusion with sources and sinks, and the fractional Fokker-Planck equation [4] for anomalous diffusion in an external field, etc. In fact, it has been shown (see, for example, [6,12,13,47,49]) that anomalous diffusion is ubiquitous in physical and biological systems where trapping and binding of particles can occur. Anomalous diffusion deviates from the standard Fichean description of Brownian motion, the main character of which is that its mean squared displacement is a nonlinear growth with respect to time, such as $\langle x^2(t)\rangle \sim t^{\alpha}$.

The space-time fractional diffusion equation (STFDE) considered in this paper is of interest not only in its own right, but also in that it constitutes the principal part in solving many other FPDEs. The physical background includes modeling turbulent flow, chaotic dynamics charge transport in amorphous semiconductors [43, 44], NMR diffusometry in disordered materials [32], and dynamics of a bead in polymer network [3]. In [33], Nigmatullin used the fractional diffusion equation to describe diffusion in media with fractal geometry. Mainardi [29] pointed out that the propagation of mechanical diffusive wave in viscoelastic media can be modeled by STFDE.

The universality of anomalous diffusion phenomenon in physical and biological experiments has led to an intensive investigation on the fractional differential equations in recent years. For example, the TFDE and related equations have been investigated in analytical and numerical frames by a number of authors [14, 26, 45, 54]. Schneider and Wyss [45] and Wyss [54] investigated the Green functions and their properties for the time fractional diffusion wave equations. Gorenflo et al. [14, 15] used the similarity method and Laplace transform to obtain the scale invariant solution of TFDE in terms of the Wright function. The work done on the numerical solution of the TFDE includes finite difference methods by Liu et al. [27], Sun and Wu [51], Langlands and Henry [20] and so on. More recently, Lin and Xu [23] proposed a finite difference scheme in time and Legendre spectral method in space for TFDE. A convergence rate of $(2-\alpha)$-order in time and spectral accuracy in space of the method was rigourously proved. In [10, 11, 41], Ervin and Roop presented a Galerkin finite element approximation for variational solution to the steady state fractional advection dispersion equations. Very recently, Li and Xu [22] proposed a time-space spectral method for TFDE based on a weak formulation, and detailed error analysis was carried out.

A suitable variational formulation is the starting point of many numerical methods, such as finite element methods and spectral methods. The existence and uniqueness of the variational solution is thus essential for these methods to be efficient. The construction of the variational formulation strongly relies on the choice of suitable spaces and norms. The main contribution of this paper includes: First, we establish the well-posedness of the weak formulation of STFDE, with the help of the introduction of suitable fractional Sobolev spaces and norms. We clearly distinguish two different definitions of the fractional derivative: Riemann-Liouville derivative and Caputo one. We find that in the case of Riemann-Liouville definition there is no need to impose any explicit initial