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Resonance Clustering in Wave Turbulent Regimes: Integrable Dynamics

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Abstract. Two fundamental facts of the modern wave turbulence theory are 1) existence of power energy spectra in k-space, and 2) existence of "gaps" in this spectra corresponding to the resonance clustering. Accordingly, three wave turbulent regimes are singled out: *kinetic*, described by wave kinetic equations and power energy spectra; *discrete*, characterized by resonance clustering; and *mesoscopic*, where both types of wave field time evolution coexist. In this review paper we present the results on integrable dynamics of resonance clusters appearing in discrete and mesoscopic wave turbulent regimes. Using a novel method based on the notion of dynamical invariant we show that some of the frequently met clusters are integrable in quadratures for arbitrary initial conditions and some others-only for particular initial conditions. We also identify chaotic behaviour in some cases. Physical implications of the results obtained are discussed.

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Key words: Resonance clustering, Hamiltonian formulation, wave turbulent regimes, NR-diagram, integrable dynamics.

1 Introduction

The broad structure of modern nonlinear science born at the edge of physics and mathematics includes an enormous number of applications in cosmology, biochemistry, electronics, optics, hydrodynamics, economics, neuroscience, etc. The emergence of nonlinear science itself as a collective interdisciplinary activity is due to the awareness that

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its dynamic concepts first observed and understood in one field (for example, population biology, flame-front propagation, non-linear optics or planetary motion) could be useful in others (such as in chemical dynamics, neuroscience, plasma confinement or weather prediction). The theory of integrable Hamiltonian systems, a generalization of the classical theory of differential equations, is the fundamental part of the whole nonlinear science for it yields good mathematical models for many physical phenomena. Various classifications of integrable systems are presently known which turned out to be quite useful for physical applications. Classifications are known based on the various intrinsic properties of integrable systems [56]: symmetries, conservation laws, Lax-pairs, etc. In [3] the general classification of integrable Hamiltonian systems is presented based on the form of their topological invariants. The usefulness of this classification is demonstrated in several problems on solid mechanics. In particular, it is proven that two famous problems-the Euler case in rigid body dynamics and the Jacobi problem of geodesics on the ellipsoid-are orbitally equivalent. In [12] the idea of classification is presented based on normal forms of a certain class of bi-hamiltonian PDEs. Miscellaneous hierarchies of integrable PDEs are presented in [50].

The list can be prolonged further but the main point for us presently is the following: the notion of integrability itself is ambitious! There are many quite different definitions of integrability, for instance integrability in terms of elementary functions (equation $\ddot{y}=-y$ has the explicit solution $y=a\sin(x+b)$); integrability *modulo* class of functions (equation $\ddot{y}=f(y)$ has general solutions in terms of elliptic functions), etc. An example of less obvious definition of integrability is C-integrability, first introduced in [6]: integrability *modulo* change of variables, meaning that a *nonlinear* equation is called C-integrable if it can be turned into a *linear* equation by an appropriate invertible change of variables. For instance, Thomas equation $\psi_{xy}+\alpha\psi_x+\beta\psi_y+\psi_x\psi_y=0$ is C-integrable. Profound discussion on the subject can be found in [36]. In the present paper, integrability is interpreted in terms of the existence of a number of independent dynamical invariants of the system; for each in-this-sense-integrable system, solutions are then written out *in quadratures*.

The dynamical systems we are interested in, describe nonlinear resonance clusters appearing in evolutionary dispersive wave systems in two space variables. Nonlinear resonances are ubiquitous in physics. They appear in a great amount of typical mechanical systems [13,38], in engineering [8,18,39,63], astronomy [55], biology [16], etc. Euler equations, regarded with various boundary conditions and specific values of some parameters, describe an enormous number of nonlinear dispersive wave systems (capillary waves, surface water waves, atmospheric planetary waves, drift waves in plasma, etc) all possessing nonlinear resonances.

The classical approach of statistical wave turbulence theory in a nonlinear wave system assumes weak nonlinearity, randomness of phases, infinite-box limit, existence of an inertial interval in wavenumber space (k_0,k_1) (where energy input and dissipation are separated in scales from both energy input and dissipation area) as well as some other assumptions omitted here (see [70] for more details). As a result, the wave system is energy conserving, and wave kinetic equations describing the wave spectrum have sta-