Fast Spectral Collocation Method for Surface Integral Equations of Potential Problems in a Spheroid

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Abstract. This paper proposes a new technique to speed up the computation of the matrix of spectral collocation discretizations of surface single and double layer operators over a spheroid. The layer densities are approximated by a spectral expansion of spherical harmonics and the spectral collocation method is then used to solve surface integral equations of potential problems in a spheroid. With the proposed technique, the computation cost of collocation matrix entries is reduced from $\mathcal{O}(M^2N^4)$ to $\mathcal{O}(MN^4)$, where N^2 is the number of spherical harmonics (i.e., size of the matrix) and M is the number of one-dimensional integration quadrature points. Numerical results demonstrate the spectral accuracy of the method.

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1 Introduction

In this paper we consider a spectral collocation method for the surface integral equations arising from three-dimensional potential problems

$$\int_S \mu(\mathbf{r}) \frac{1}{|\mathbf{r}' - \mathbf{r}|} d\mathbf{r} = f(\mathbf{r}'), \tag{1.1}$$

$$\int_S \rho(\mathbf{r}) \frac{\partial}{\partial n_\mathbf{r}} \left(\frac{1}{|\mathbf{r}' - \mathbf{r}|} \right) d\mathbf{r} = g(\mathbf{r}'), \tag{1.2}$$

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where f and g are given, $\mathbf{r},\mathbf{r}' \in S$ and S is the surface of a three-dimensional spheroid defined by

$$S = \left\{ \mathbf{r} = (x,y,z) \ \Big| \ \frac{x^2 + y^2}{R^2} + \frac{z^2}{L^2} = 1 \right\}. \tag{1.3}$$

The surface is called a prolate spheroid if $L > R$, and an oblate spheroid if $L < R$, and if $L = R$ it is a sphere. In Eqs. (1.1) and (1.2), the unknown scalar functions $\mu(\mathbf{r})$ and $\rho(\mathbf{r})$ are called single and double layer density functions, respectively, and n is the unit inner normal direction at a field point $\mathbf{r}$ on S. The potential problems arise from many fields of physics and engineering [16] through boundary integral formulations of differential equations of elliptic-type. In particular, in biological applications, spheroidal geometries are often used to model some types of molecule in the study of electrostatics [7] and the free energy [2].

There are many issues [5, 19] in accurate and efficient numerical approximations of potential problems, including the choices of basis functions, the treatment of singularities and fast solvers for the resulting discrete algebraic systems. For problems defined on a smooth boundary, it is natural to consider spectral methods [6, 18] due to its infinite-order convergence, so-called the spectral convergence. The spectral methods have been applied for numerical solutions of problems in spheroidal geometries, for example, in geodesy and in global atmospheric models [9, 17], by using the spherical harmonics as the basis functions. This is due to the fact that the spherical harmonics offer nice properties in approximating spherical functions with exponential convergence and the ease in handling pole singularities [6]. For the boundary integral equations considered in this paper, the spherical harmonics have been used by Atkinson [3,4], Graham and Sloan [12], Chen [8], and Ganesh [10] to construct spectral methods for three-dimensional potential problems. Also, double Fourier series have found applications in solving potential problems [11].

For spectral methods of surface integral equations, the calculation of collocation coefficients (or Galerkin coefficients in Galerkin methods) remains one of the computational challenges as numerical quadratures over the full boundary surface is required in calculating each entry of the coefficient matrix. For example, if N^2 basis functions and M^2-point two-dimensional quadratures are used, the total operations of order $\mathcal{O}(M^2 N^4)$ are required to produce the coefficient matrix of a collocation method. Here M can become large with the increase of basis function order N in order to maintain the high-frequency information from the higher order basis functions. Even with relatively small N, the cost can be still high. In this paper, we will propose a new technique which will reduce the two-dimensional integral to one dimensional one. This is made possible by an analytical formula along the longitudinal direction for the spheroidal surface with the help of a hypergeometric function. The hypergeometric function can be calculated fast even at singular points thanks to linear transformation formulas. As a result, the complexity of filling up the collocation matrix is then reduced to the order of $\mathcal{O}(MN^4)$. With some generalization, this technique can be extended to boundary element methods and artificial