

# Implicit Unstructured-Mesh Method for Calculating Poiseuille Flows of Rarefied Gas

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**Abstract.** An implicit high-order accurate method for solving model kinetic equations is proposed. The method is an extension of earlier work on the construction of an explicit TVD method for hybrid unstructured meshes in physical space and is illustrated on the Poiseuille flow of rarefied gas. Examples of calculations are provided for different Knudsen numbers and mesh resolutions, which illustrate the efficiency and high accuracy of the new scheme.

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**Key words:** Poiseuille, kinetic, unstructured, mixed-element, micro, rarefied, implicit.

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## 1 Introduction

The numerical solution of the Boltzmann kinetic equation with the exact or model collision integral is a major technique for studying rarefied gas flows. Usually, the numerical methods for this equation use structured meshes [10,11,28] and cannot be readily applied to flows in complex spatial configurations. Recently, high-order accurate finite-volume methods on unstructured meshes in physical space have been proposed. These include a Cartesian Cut Cell method for the Boltzmann equation with the exact collision integral [8] and an unstructured-mesh scheme [24] for the S-model kinetic equation [17, 18] in two spatial dimensions. In particular, the method [24] consists of two main components: an explicit high-order Total Variation Diminishing (TVD) scheme for discretising the advection operator on hybrid unstructured meshes and a conservative procedure for computing macroscopic gas parameters [23, 26]. It is thus applicable to gas flows in arbitrarily shaped two-dimensional spatial domains and across the whole range of

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Knudsen numbers from the free-molecular to continuum regimes. The method has been recently adopted to pressure-driven isothermal flows of rarefied gas in long micro channels (Poiseuille flow) [25], which represent an important class of rarefied gas dynamics problems [19,20]. Computational results were given for channels with circular, triangular and polygonal cross-sectional areas, thus showing the applicability of the unstructured-mesh approach in the context of rarefied gas flows.

All mentioned unstructured-mesh methods use explicit time discretisation methods. They are therefore simple to implement, but also computationally expensive as compared to implicit structured-mesh alternatives [11, 26, 28]. For transitional and nearly-continuum flows the stability condition of an explicit method results in a small time step, proportional to the spatial cell size in thin Knudsen layers. As a result, the computational cost of computing near-continuum steady-state solutions using explicit methods may be prohibitively high.

The goal of the present work is to develop an implicit version of the method [24,25] for computing steady-state solutions of kinetic equations on two-dimensional unstructured meshes. For the sake of simplicity, the idea is explained for the linearised kinetic equation as applied to Poiseuille flows. On unstructured meshes, the use of the implicit time discretisation results in a large sparse system of linear equations for the values of the distribution function at the new time level. The direct solution of the system is expensive and requires the use of a large computer memory. To reduce the computational cost, an approximate factorisation based on [13, 14] is used. The resulting numerical method is almost as efficient per time step as the one-step explicit scheme with the same reconstruction procedure. However, it allows to use time step at least an order of magnitude larger as compared to the original explicit scheme. As a result, the overall computational cost of calculating the steady-state solutions is significantly reduced.

The paper is organised as follows. In Section 2 the flow problem and governing equations are formulated. In Section 3 the explicit method on mixed-element unstructured meshes [24,25] is briefly outlined. The implicit time discretisation scheme is described in detail in Section 4. In Section 5 numerical results are presented for the circular pipe flow and compared with those published in the literature. Conclusions are drawn in Section 6.

## 2 Problem formulation

Consider low-speed stationary flows of a monatomic rarefied gas from reservoir 1 to reservoir 2 through a channel with an arbitrary but constant cross-section  $A$  and finite length  $2L$ . Inside the reservoirs away from the channel the gas is at rest with pressures  $p_1 < p_2$  and with equal temperatures  $T_1 = T_2$ . Let us introduce a Cartesian coordinate system  $(x, y, z)$ , in which the  $Oz$  axis is directed along the channel. The centre of the coordinate system is located in the middle of the channel  $z=0$ . The complete accommodation of momentum and energy of molecules occurs at the channel walls, which are kept under the constant temperature  $T_w = T_1 = T_2$ . It is further assumed that the channel length