Commun. Comput. Phys.
doi: 10.4208/cicp.090909.100310a

Vol. **8**, No. 5, pp. 995-1015
November 2010

On Exact Conservation for the Euler Equations with Complex Equations of State

J. W. Banks*

Center for Applied Scientific Computing, Lawrence Livermore National Laboratory, Livermore, CA 94551, USA.

Received 9 September 2009; Accepted (in revised version) 10 March 2010

Available online 31 May 2010

Abstract. Conservative numerical methods are often used for simulations of fluid flows involving shocks and other jumps with the understanding that conservation guarantees reasonable treatment near discontinuities. This is true in that convergent conservative approximations converge to weak solutions and thus have the correct shock locations. However, correct shock location results from any discretization whose violation of conservation approaches zero as the mesh is refined. Here we investigate the case of the Euler equations for a single gas using the Jones-Wilkins-Lee (JWL) equation of state. We show that a quasi-conservative method can lead to physically realistic solutions which are devoid of spurious pressure oscillations. Furthermore, we demonstrate that under certain conditions, a quasi-conservative method can exhibit higher rates of convergence near shocks than a strictly conservative counterpart of the same formal order.

AMS subject classifications: 65N08, 35L60, 35L65, 76N15

Key words: Euler equations, complex EOS, JWL EOS, Godunov methods.

---

1 Introduction

The use of conservative schemes for simulations of solutions to hyperbolic conservation laws in the presence of discontinuities is ubiquitous. The reasons draw largely from the fact that convergent conservative schemes are known to converge to weak solutions in the presence of shocks by the Lax-Wendroff theorem [1]. On the other hand, non-conservative schemes can converge to non-weak solutions which violate the integral conservation equations [2]. However, weak solutions which violate physical properties, such as positivity of density or entropy satisfaction, are equally as troublesome as non-weak solutions and there is very little theory guaranteeing convergence for nonlinear systems to the appropriate "vanishing viscosity" solution (one notable exception is the random

*Corresponding author. *Email address:* `banks20@llnl.gov` (J. W. Banks)

choice method of Glimm [3]). It is interesting then that so much emphasis has been placed on exact conservation rather than rapid convergence to the relevant vanishing viscosity solution.

One prototypical physical system described by hyperbolic conservation laws is that of gas dynamics. Here the governing system is the Euler equations and for cases where the equation of state is sufficiently nonlinear, issues relating to exact conservation rise very much to the surface. The classical realization of these problems is unphysical oscillations, particularly in the pressure, arising near material interfaces in multi-material flows [4–10]. Somewhat less appreciated is the fact that the same type of behavior can be exhibited for single component flows with complicated equations of state [11].

The multi-component case has been studied extensively and many non-conservative or quasi-conservative schemes have been proposed and used to great effect in that context. Nonlinearities in the equation of state result when sharp material interfaces are smeared through the use of a capturing scheme. To remedy this, the paper by Quirk and Karni [5] for instance, relies on a non-conservative primitive formulation and adds source terms which restore conservation where possible. Other choices that have been made include the use of a hybrid approach that relies on the conservative update whenever possible but uses the primitive update near interfaces [4], conservative approximations which add source terms to break conservation as required near material interfaces [9,10], and special advection rules for particular components of the equation of state [6,7]. All of these approaches seek to realize physically realistic treatment of the material interface through the use of non-conservative schemes and rely on the fact that the poor behavior exhibited by capturing schemes is limited to a small region about the material interface.

In [11] the authors show that poor behavior is not limited to material interfaces and problems can occur for single component flows with sufficiently nonlinear equations of state. In that paper, the authors present a new numerical method to treat such flows. Their scheme relies on a specific class of Riemann solver, a specific equation of state, and in the end introduces special advection rules for the adiabatic exponent which amounts to a modification of the equation of state. Their approach does seems to have some efficacy when it is applicable, but alterations of the equation of state can lead to inconsistent numerical schemes and great care must be used. Without assurances that the numerical system is consistent with the original PDE as well as convergent, this scheme also faces obstacles in terms of application of the Lax-Wendroff theorem.

The state of matters then seems to be that the classical, that is to say consistent and conservative, schemes may produce physically unrealistic results; especially near contacts. As a result, even though some theoretical benefit is achieved through the use of fully conservative schemes, they can produce approximations which are not useful and so modifications to these schemes are developed. On one hand there is the class of schemes which modify the governing equations and thus risk inconsistency [11]. On the other hand there is the class of schemes which modify conservation with the obvious associated risks [4–10]. It is not at all clear that these approaches are entirely different and it may be the case that the actual numerical approximations are quite similar.