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A New Stable Algorithm to Compute Hankel Transform Using Chebyshev Wavelets

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Abstract. A new stable numerical method, based on Chebyshev wavelets for numerical evaluation of Hankel transform, is proposed in this paper. The Chebyshev wavelets are used as a basis to expand a part of the integrand, $rf(r)$, appearing in the Hankel transform integral. This transforms the Hankel transform integral into a Fourier-Bessel series. By truncating the series, an efficient and stable algorithm is obtained for the numerical evaluations of the Hankel transforms of order $\nu > -1$. The method is quite accurate and stable, as illustrated by given numerical examples with varying degree of random noise terms $\varepsilon\theta_i$ added to the data function $f(r)$, where θ_i is a uniform random variable with values in $[-1,1]$. Finally, an application of the proposed method is given for solving the heat equation in an infinite cylinder with a radiation condition.

AMS subject classifications: 65R10

Key words: Hankel transforms, Bessel functions, Chebyshev wavelets, random noise term.

1 Introduction

There are several integral transforms which are frequently used as a tool for solving numerous scientific problems. It is well known that the Fourier transform (FT) is used to obtain spatial spectrum of optical light [1]. Fourier optics is widely used in optical instrument design, optical propagation through lenses and in quadratics graded index mediums. Most classical optical systems like mirrors or lenses are axially symmetrical

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devices. In many practical problems, data are often acquired in such a form that is desirable to perform a two-dimensional polar Fourier transform that is a Hankel transform (HT) rather than the Cartesian forms. So, we transform the Cartesian coordinates into the polar coordinates.

Let $f(x,y)$ be an input field such that it can be separated as $f(x,y)=f_1(x)f_2(y)$, where f_1 and f_2 are independent functions. Then its two-dimensional Fourier transform $\hat{f}$ is also separable as the same symmetry property is transposed through a linear FT. Hence, $\hat{f}(u,v)=\hat{f}_1(u)\cdot\hat{f}_2(v)$.

Changing to the polar coordinates and if $f(r,\theta)=f(r)$ is axially symmetrical, then in [2], it was shown that

$$\hat{f}(k,\varphi)=\frac{1}{2}\int_0^\infty d(r^2)f(r)J_0(kr)\equiv F_0(k), \tag{1.1}$$

which is also axially symmetrical in the Fourier frequency domain, where F_0 is the Hankel transform of order zero. The general Hankel transform pair with the kernel being J_ν is defined as

$$F_\nu(p)=\int_0^\infty rf(r)J_\nu(pr)dr, \tag{1.2}$$

and HT being self reciprocal, its inverse is given by

$$f(r)=\int_0^\infty pF_\nu(p)J_\nu(pr)dp, \tag{1.3}$$

where J_ν is the νth-order Bessel function of first kind.

The Hankel transform arises naturally in the discussion of problems posed in cylindrical coordinates and hence, as a result of separation of variables, involving Bessel functions. The Hankel transform is frequently used as a tool for solving numerous scientific problems. It is widely used in several fields like, elasticity [4], optics [5,6], fluid mechanics [7], seismology [8], astronomy and image processing [9–16]. The Hankel transform becomes very useful in analysis of wave fields where it is used in mathematical handling of radiation, diffraction, and field projection. Recently, it has been utilized to study pseudo-differential operators. Singh and Pandey [17] used HT of order ν, $\nu\in\mathbf{R}$ to study a special class of pseudo-differential operator (PDO) $(-x^{-1}D)^\nu$, $D=d/dx$ and proved that the (PDO) is almost an inverse of HT operator h_ν in the sense that

$$h_\nu o(-x^{-1}D)^\nu(\varphi)=h_0(\varphi)$$

over certain Freshet space F, thus representing the PDO as a Fourier-Bessel series. Further, in 1995, Singh [18], using the HT representation of the PDO, proved that $e^{-\alpha x^2}$, $\mathrm{Re}\alpha>0$ are the eigenfunctions and $e^{-x^2/2}$ is a fixed point of $(-x^{-1}D)^\nu$, $\nu\in\mathbf{C}$.