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An Efficient Algorithm to Simulate a Brownian Motion Over Irregular Domains

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Abstract. In this paper, we present an algorithm to simulate a Brownian motion by coupling two numerical schemes: the Euler scheme with the random walk on the hyper-rectangles. This coupling algorithm has the advantage to be able to compute the exit time and the exit position of a Brownian motion from an irregular bounded domain (with corners at the boundary), and being of order one with respect to the time step of the Euler scheme. The efficiency of the algorithm is studied through some numerical examples by comparing the analytical solution with the Monte Carlo solution of some Poisson problems. The Monte Carlo solution of these PDEs requires simulating Brownian motions of different types (natural, reflected or drifted) over an irregular domain.

AMS subject classifications: 60G50, 60G40, 65C05, 65C30, 65C20

Key words: Brownian motion, Monte Carlo methods, partial differential equations, Euler scheme, random walk on rectangles.

1 Introduction

The term "Brownian motion" is used to describe a particle in a fluid that has a permanent and random agitation. The usage of this expression has been extended to all the mathematical models that describe random phenomena such that the fluctuations of the option markets in finance. Under appropriated hypothesis, a large class of stochastic processes can be modeled using the Brownian motion.

Mathematically, the Brownian motion is a Wiener process in which the conditional transition density of a particle at instant $t+\Delta t$ given its position x at instant t is a Gaussian law with mean x and variance Δt. Let X_t be this stochastic process in dimension d. When

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the Euler scheme is used to simulate X_t over the time interval $[0,T]$ using N time steps, X_t is discretized by the sequence of random variables $\tilde{X}_{t_k}$ as follows:

$$\begin{cases} \tilde{X}_0 = x, \\ \tilde{X}_{t_k} = \tilde{X}_{t_{k-1}} + \sqrt{\Delta t} W_k, \end{cases} \tag{1.1}$$

where $\Delta t = T/N$, $\tilde{X}_{t_k} \overset{(law)}{=} X_{k\Delta t}$ is an approximation of the position of the particle at time t_k, W_k is a vector of d Gaussian variables with unit variance and $k=1,\cdots,N$.

We are interested in simulating X_t over a bounded regular domain $D \subset \mathbb{R}^d$. In this case, two random variables are associated to X_t, the exit time τ defined by:

$$\tau = \inf\{t, X_t \notin D\}$$

and the exit position X_τ, the position the particle on the boundary of D at time τ. The exit time is approximated by $\tau_d = \inf\{t_k, \tilde{X}_{t_k} \notin D\}$. It is known that τ_d is not a good approximation of τ because it is of order $1/2$ with respect to Δt (see [1]). A better approximation of τ would be to consider at each iteration k of the Euler scheme the exit time of the continuous stochastic process $\tilde{X}_t$ defined between two iterations by:

$$\text{for } \ t \in [k\Delta t, (k+1)\Delta t], \quad \tilde{X}_t = \tilde{X}_{t_k} + \sqrt{t - k\Delta t} W_k, \tag{1.2}$$

and $\tau_c = \inf\{t, \tilde{X}_t \notin D\}$. τ_c is the exit time of the continuous time stochastic process $\tilde{X}_t$ starting at $\tilde{X}_{t_k}$. Its estimation is independent from the time step Δt. Some algorithms were presented to compute τ_c in [4,9] for the one-dimensional case, in [1,4,5] from a regular domain, [14] for a bi-dimensional cone and more generally in [3] in the case of a reflected Brownian motion. These algorithms are based on considering the Brownian bridge between $\tilde{X}_{t_k}$ and $\tilde{X}_{t_{k+1}}$ and computing the probability that the Brownian bridge intersects the boundary. These algorithms have the drawback of assuming that the boundary is locally smooth when the Brownian particle is near the boundary. Hence these algorithms cannot consider a rectangular domain (for example) and simulate the Brownian motion near the corners.

In this paper, we propose to simulate the exit time τ and the exit position X_τ from a bounded domain, by coupling the Euler scheme with the random walk on the hyper-rectangles. The advantage of this algorithm over the ones described in [1,4,5] is to handle domain boundaries with corners and for different types of Brownian motions (natural, reflected, drifted). To show the efficiency of our algorithm, numerical experiments are carried out by comparing the analytical with the Mont Carlo solution of some Poisson problems. The Monte Carlo solution is computed using the Monte Carlo method which requires the simulation of the exit time of the Brownian motion.

2 Brownian motion and diffusion process

To describe this stochastic process, the transition density $p(x,y,t)$ of the particle's position at position y and time t starting at x must be specified. In the following, we show how it