

## Adaptivity and A Posteriori Error Control for Bifurcation Problems I: The Bratu Problem

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Received 29 July 2009; Accepted (in revised version) 12 February 2010

Available online 17 May 2010

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**Abstract.** This article is concerned with the numerical detection of bifurcation points of nonlinear partial differential equations as some parameter of interest is varied. In particular, we study in detail the numerical approximation of the Bratu problem, based on exploiting the symmetric version of the interior penalty discontinuous Galerkin finite element method. A framework for *a posteriori* control of the discretization error in the computed critical parameter value is developed based upon the application of the dual weighted residual (DWR) approach. Numerical experiments are presented to highlight the practical performance of the proposed *a posteriori* error estimator.

**AMS subject classifications:** 37G10, 37M20, 65N12, 65N15, 65N30

**Key words:** Bifurcation theory, Bratu problem, *a posteriori* error estimation, adaptivity, discontinuous Galerkin methods.

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## 1 Introduction

Understanding the nature of solutions to nonlinear partial differential equations (PDEs) remains one of the greatest challenges in modern scientific computing. Some fundamental questions include: "How many solutions exist as some parameter of interest is varied?"; "Are the solutions linearly stable?"; and "At what critical parameter value does a bifurcation occur?". In this article we consider the latter question and in particular address the issue concerning the accuracy of the computed critical value by means of

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*a posteriori* error estimation. For this purpose we investigate the Bratu problem, see, for example, Wazwaz [35], which can be viewed as a model of some phenomenon exhibiting diffusion with exponential growth. Although the Bratu problem is essentially of academic interest, it serves as an excellent model situation in which to demonstrate the computational approach developed in this article, as it contains many of the key features inherent in the study of more general nonlinear PDEs of practical interest.

In the numerical study of nonlinear PDEs, the estimation of the critical parameter at which a bifurcation may occur can be performed by discretizing a suitable extended system of PDEs; see, for example, Seydel et al. [30, 31] and Moore and Spence [28]. In essence, this process involves determining the parameter value and associated solution at which the Jacobian of the underlying nonlinear PDE has a zero eigenvalue. For the discretization of the extended system we propose to exploit the symmetric version of the interior penalty discontinuous Galerkin (DG) method; see, for example, [2] where a unified analysis of a number of DG methods is presented. Our use of a DG method is primarily due to the benefits in terms of the ease of implementation of automatic mesh adaptation procedures.

Over the past few decades, tremendous progress has been made in the area of *a posteriori* error estimation and adaptive finite element approximation of partial differential equations; for a review of some of the main developments in the subject we refer to the recent monographs [1, 32, 34], and the articles [5, 15]. Despite a number of significant advances in the field, much of the research to date has focused on source problems. In the context of eigenvalue error estimation for determining whether a solution to a PDE is linearly stable or not, we mention the recent articles [13, 14, 25, 29] for the finite element approximation of second-order self-adjoint elliptic eigenvalue problems. For related work, based on considering the eigenvalue problem as a parameter-dependent nonlinear equation, see Verfürth [33, 34], for example, while convergent adaptive algorithms for eigenvalue problems have been analysed in [7, 17]. More recently, in the article [11], we considered the *a posteriori* estimation of the error in the leading eigenvalue for the hydrodynamic stability problem. In particular, we employed a dual weighted residual (DWR) *a posteriori* error estimator, see [4, 16, 21], for example, specifically tailored to assess the accuracy of the computed leading eigenvalue. Here, the discretization error stemming from both the numerical approximation of the steady incompressible Navier-Stokes equations, as well as the error arising from the approximation of the corresponding eigenvalue problem itself was controlled. The purpose of this article is to consider the natural extension of these ideas to bifurcation problems. More precisely, we derive computable *a posteriori* bounds on the error in the DG approximation of the critical parameter value for the Bratu problem, based on exploiting the general DWR methodology. Additionally, we extend the ideas presented in Moore & Spence [28] to develop an efficient solution algorithm for both the underlying primal and dual problems to the DG setting.

Rigorous proofs of the existence of bifurcation points in continuous systems, such as the Bratu problem in 2 or more dimensions, are extremely difficult. Indeed one of the primary motivations for developing numerical methods for such problems is that they