Commun. Comput. Phys.
doi: 10.4208/cicp.030210.051110a

Vol. **10**, No. 2, pp. 339-370
August 2011

Convergent Adaptive Finite Element Method Based on Centroidal Voronoi Tessellations and Superconvergence

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Received 3 February 2010; Accepted (in revised version) 5 November 2010

Communicated by Pingwen Zhang

Available online 27 April 2011

Abstract. We present a novel adaptive finite element method (AFEM) for elliptic equations which is based upon the Centroidal Voronoi Tessellation (CVT) and superconvergent gradient recovery. The constructions of CVT and its dual Centroidal Voronoi Delaunay Triangulation (CVDT) are facilitated by a localized Lloyd iteration to produce almost equilateral two dimensional meshes. Working with finite element solutions on such high quality triangulations, superconvergent recovery methods become particularly effective so that asymptotically exact a posteriori error estimations can be obtained. Through a seamless integration of these techniques, a convergent adaptive procedure is developed. As demonstrated by the numerical examples, the new AFEM is capable of solving a variety of model problems and has great potential in practical applications.

AMS subject classifications: 65N50, 65N15

Key words: Finite element methods, superconvergent gradient recovery, Centroidal Voronoi Tessellation, adaptive methods.

1 Introduction

Adaptive finite element methods (AFEM) have been widely studied for over two decades and are now standard tools in numerical simulations of scientific and engineering prob-

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lems [1,2,28,35,45]. AFEMs are especially attractive for problems with solutions which are of singular or multi-scale nature [1,3,8,10,38]. A standard adaptive finite element method for solving a partial differential equation (PDE) consists of iterations between several key components: *SOLVE* → *ERROR ESTIMATE* → *SIZING MODIFY* → *REFINE/COARSEN*. More specifically, the procedure *SOLVE* solves for the discrete finite element solution of the PDE on the current mesh; the procedure *ERROR ESTIMATE* performs a posteriori error estimation over the computed finite element solution; with the computed a posteriori estimate, the procedure *SIZING MODIFY* introduces the new mesh sizing based on an error equal distribution principle; and the procedure *REFINE /COARSEN* changes the mesh through moving and/or inserting/removing vertices together with other compatible element modifications. The convergence of an adaptive finite element algorithm implies that, starting from a given coarse mesh, the adaptation loop converges within a prescribed error tolerance in a finite number of iterations.

To develop a robust and convergent adaptive finite element method for elliptic problems on a complicated geometry, it is necessary to construct reliable a posteriori error estimates and effective element marking (for refinement/coarsening) strategy, along with suitable mesh sizing modification and effective mesh refinement and optimization. While various techniques have been developed to address each of the above issues, we present a new approach in this work for the adaptive finite element solution of two dimensional elliptic equations. The main ideas underneath our method consist of the use of superconvergence properties of the finite element solutions based on Centroidal Voronoi Delaunay triangulations [14–18,27] for the derivation of asymptotically exact a posteriori error estimations, and the use of localized Lloyd iterations [34] for efficient high quality meshing. Moreover, we show that these ideas can be seamlessly and systematically integrated into a successful and convergent adaptive finite element algorithm.

Let us first provide some brief discussions on several key components of AFEM and review some existing works. We note that given the large literature on the subject, our discussion is very limited and only some most relevant works to our approach are mentioned.

First, among the various a posteriori error estimation approaches, the residual-based and recovery-type methods have been widely accepted [1,4,36,43,45]. A particularly popular method is the ZZ-SPR approach, which is based on a local least squares fitting and has been widely used in numerical engineering practices, especially in commercial softwares, due to its robustness in effective a posteriori error estimates and its efficiency in computer implementation [45]. It is a common belief that the robustness of the Z-Z technique is rooted in its superconvergence property under structured or mildly structured meshes [4,36,43], which can be generated using many existing mesh generators [14,15,23,41]. In [43], the effectiveness of the ZZ-SPR technique for linear triangular elements was established provided that the mesh satisfies the strong regularity or the quadrilateral parallelogram property. The construction of a triangular mesh with such a property and a given sizing specification has also been well documented. If such superconvergence properties hold for meshes in an adaptive procedure, one can then expect to