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Simulating an Elastic Ring with Bend and Twist by an Adaptive Generalized Immersed Boundary Method

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Abstract. Many problems involving the interaction of an elastic structure and a viscous fluid can be solved by the immersed boundary (IB) method. In the IB approach to such problems, the elastic forces generated by the immersed structure are applied to the surrounding fluid, and the motion of the immersed structure is determined by the local motion of the fluid. Recently, the IB method has been extended to treat more general elasticity models that include both positional and rotational degrees of freedom. For such models, force and torque must both be applied to the fluid. The positional degrees of freedom of the immersed structure move according to the local linear velocity of the fluid, whereas the rotational degrees of freedom move according to the local angular velocity. This paper introduces a spatially adaptive, formally second-order accurate version of this generalized immersed boundary method. We use this adaptive scheme to simulate the dynamics of an elastic ring immersed in fluid. To describe the elasticity of the ring, we use an unconstrained version of Kirchhoff rod theory. We demonstrate empirically that our numerical scheme yields essentially second-order convergence rates when applied to such problems. We also study dynamical instabilities of such fluid-structure systems, and we compare numerical results produced by our method to classical analytic results from elastic rod theory.

AMS subject classifications: 65M06, 65M50, 74S20, 76D05

Key words: Immersed boundary method, Kirchhoff rod theory, adaptive mesh refinement.

1 Introduction

In the 1970's, Peskin developed the immersed boundary (IB) method to study the fluid dynamics of heart valves [1, 2]. Since then, the IB method has become widely used for

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simulating dynamic problems of fluid-structure interaction in which an elastic structure is immersed in a viscous incompressible fluid [3]. The IB method uses Lagrangian variables to describe the motion of the elastic structure and Eulerian variables to describe the motion of the fluid. In the conventional IB formulation of such problems, the elastic structure applies forces to the fluid that generally act to alter the fluid motion, and the structure moves according to the local velocity of the fluid.

More recently, a generalization of the IB method was introduced to study the dynamics of elastic rods that are modeled using a version of Kirchhoff rod theory [4,5]. Kirchhoff rod theory describes the force and torque generated by an elastic rod in terms of the position of its center line together with a Lagrangian field of orthonormal director vectors that are attached to that center line. These director vectors are rotational degrees of freedom that account for the bending and twisting of the rod. Such representations are useful for describing the motion of filamentous structures like DNA and cables [6–8]. To couple such generalized elasticity models to the fluid within the IB framework, it was necessary to extend the IB formulation to treat elastic structures that are represented in terms of both positional and rotational degrees of freedom. The key features of this generalized IB method are that it applies both the force and the torque generated by the elastic rod to the fluid, and that the elastic rod moves according to both the local linear and angular velocities of the surrounding fluid. Specifically, the local linear velocity determines the motion of the center line, and the local angular velocity determines the rotation of the orthonormal triad of director vectors attached to the structure. So far, this generalized IB method has been used exclusively for problems involving the dynamics of elastic rods immersed in fluid; however, this generalized IB framework is not restricted to such structural models, and it may ultimately find use in coupling other elasticity models that include both positional and rotational degrees of freedom, such as elastic shells, to a surrounding fluid.

The original generalized IB method employed a uniform discretization of the equations of motion that was only first-order accurate [4, 5]. In this work, we introduce an adaptive, formally second-order accurate version of the method. Our numerical scheme is based on a discretization approach previously adopted in adaptive versions of the conventional IB method [9–13]. We use a staggered-grid (i.e., maker-and-cell or MAC [14]) version of the IB method, in which the Eulerian fluid pressure is approximated at the centers of the cells of a locally refined Cartesian grid, and in which the normal components of the Eulerian fluid velocity field are approximated at the centers of the faces of the Cartesian grid cells. Our adaptive three-dimensional discretization is therefore similar, but not identical, to the two-dimensional adaptive version of the conventional IB method described by Roma et al. [9], and to the adaptive three-dimensional IB method described by Griffith [13]. As discussed by Griffith [15], staggered-grid IB methods appear to have clear advantages in terms of volume conservation and resolution of pressure discontinuities when compared to collocated IB discretizations, such as those used in earlier cell-centered adaptive IB methods [10–12].

We assess the convergence properties of our adaptive method via an empirical con-