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Quasi-Optimized Overlapping Schwarz Waveform Relaxation Algorithm for PDEs with Time-Delay

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Abstract. Schwarz waveform relaxation (SWR) algorithm has been investigated deeply and widely for regular time dependent problems. But for time delay problems, complete analysis of the algorithm is rare. In this paper, by using the reaction diffusion equations with a constant discrete delay as the underlying model problem, we investigate the convergence behavior of the overlapping SWR algorithm with Robin transmission condition. The key point of using this transmission condition is to determine a free parameter as better as possible and it is shown that the best choice of the parameter is determined by the solution of a min-max problem, which is more complex than the one arising for regular problems without delay. We propose new notion to solve the min-max problem and obtain a quasi-optimized choice of the parameter, which is shown efficient to accelerate the convergence of the SWR algorithm. Numerical results are provided to validate the theoretical conclusions.

AMS subject classifications: 30E10, 65M12, 65M55

Key words: Schwarz method, waveform relaxation, time delay, min-max problem.

1 Introduction

In the past decade, Schwarz waveform relaxation (SWR) algorithm has received much attention by many authors. The algorithm is characterised by firstly partitioning the space domain into several overlapping subdomains, and then solving the subproblems simultaneously inside each subdomain through iterations; hence the algorithm is different from

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the classical domain decomposition method [2, 3, 30] and takes the form of waveform relaxation iteration (see [23–25, 31] and references therein). We refer to [9, 13, 14] and [12] for the original idea of this algorithm.

Duo to the excellent capability in parallel computation for PDEs, the SWR algorithm is becoming more and more popular, particularly in the field of solving time dependent problems. It is a common point that the algorithm can be classified into two categories depending on the used transmission condition between the subdomains: the classical SWR algorithm and the optimized one. For the former, Dirichlet condition is used as transmission condition (see, e.g., [6, 8, 9, 11–15, 28]) and in this case the overlap between adjacent subdomains is essentially important to guarantee the convergence and the convergence rate can not be adjustable. It has been pointed out by Gander and Halpern [16] that Dirichlet condition is ineffective transmission condition. In particular, this transmission condition inhibits information exchange between subdomains and therefore the convergence rate of the classical SWR algorithm is slow.

To overcome this drawback, many authors utilize the transmission condition of Robin type on the artificial boundary interfaces (see, e.g., [1, 4, 16, 19, 21, 22, 26, 27, 32]). A free parameter, say p, is usually involved in this transmission condition and can be optimized technically to speed up the convergence of the algorithm. With the best p, the algorithm is called *optimized* SWR algorithm (for more details, we refer the interested reader to the systematic work by Gander and his colleagues [1,4,5,16–19,21,22,26,27,29], particularly to [1] and [16] for deep investigation of the optimization procedures for determining the best parameters). Nowadays, the optimized SWR algorithm is becoming more and more popular in the field of scientific and engineering computing and is adopted to solve complex problems, such as ferro-magnetics equations in the micro-magnetic model [10], wave equations [17,20], shallow-water problem [27,29] and Maxwell's equations [5], etc.

However, all of the aforementioned results are obtained for the regular PDEs without time delay. For PDEs with delay, the situation becomes very complex and the concrete results are rare. For example, the superlinear convergence of the classical SWR algorithm can be easily obtained for the regular reaction diffusion equations by using an existing results about inverse Fourier transform (see, e.g., [6, 12, 16]), while it is difficult and still unknown when time delay is taken into account. In the seminal paper [32], Vandewalle and Gander have shown that, the techniques proposed for the regular PDEs can not be straightforwardly applied to PDEs with time delay. In that paper, two representative model problems are studied: the heat equation with a constant discrete delay and one with a distributed delay. For the classical SWR algorithm, by using an elementary but very technical method they presented an estimate of the convergence rate. In our previous paper [34], we further analyzed the classical SWR algorithm for delay problems, where the reaction diffusion equations with a constant discrete delay were considered as the underlying model problems. We obtained a much sharper bound of the convergence rate and investigated the convergence behavior of the algorithm at semi-discrete level and in the case of arbitrary number of subdomains.

For the transmission condition of Robin type, the main difficulty arises from the com-