Commun. Comput. Phys.
doi: 10.4208/cicp.100412.041112a

Vol. **14**, No. 3, pp. 621-638
September 2013

On the Construction of Well-Conditioned Hierarchical Bases for $\mathcal{H}(\mathbf{div})$-Conforming $\mathbb{R}^n$ Simplicial Elements

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Received 10 April 2012; Accepted (in revised version) 4 November 2012

Available online 25 January 2013

Abstract. Hierarchical bases of arbitrary order for $\mathcal{H}(\mathbf{div})$-conforming triangular and tetrahedral elements are constructed with the goal of improving the conditioning of the mass and stiffness matrices. For the basis with the triangular element, it is found numerically that the conditioning is acceptable up to the approximation of order four, and is better than a corresponding basis in the dissertation by Sabine Zaglmayr [High Order Finite Element Methods for Electromagnetic Field Computation, Johannes Kepler Universität, Linz, 2006]. The sparsity of the mass matrices from the newly constructed basis and from the one by Zaglmayr is similar for approximations up to order four. The stiffness matrix with the new basis is much sparser than that with the basis by Zaglmayr for approximations up to order four. For the tetrahedral element, it is identified numerically that the conditioning is acceptable only up to the approximation of order three. Compared with the newly constructed basis for the triangular element, the sparsity of the mass matrices from the basis for the tetrahedral element is relatively sparser.

AMS subject classifications: 65N30, 65F35, 65F15

Key words: Hierarchical bases, simplicial $\mathcal{H}(\mathbf{div})$-conforming elements, matrix conditioning.

1 Introduction

In this paper we are concerned with the construction of well-conditioned hierarchical bases of arbitrary order for $\mathcal{H}(\mathbf{div})$-conforming $\mathbb{R}^n$ simplicial elements. Such bases are useful with the mixed finite element method [1,2] for the second-order elliptic problems,

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and may be applied with the conforming element for the numerical study of elasticity [3], electromagnetism [4], incompressible fluid flow [5], and magnetohydrodynamics [6].

In 1975 Raviart and Thomas [7] introduced the $\mathcal{H}(\mathbf{div})$-conforming triangular element for the mixed finite element method to solve the Poisson equation with zero Dirichlet boundary condition. Among other results, Nédélec [8] generalized the idea of Raviart and Thomas, and constructed the $\mathcal{H}(\mathbf{div})$-conforming tetrahedral element for the mixed finite element method, the so-called Nédélec element of the first kind. A totally different $\mathcal{H}(\mathbf{div})$-conforming tetrahedral element, the so-called Nédélec element of the second kind was constructed in 1986 [9]. Using different techniques Brezzi and collaborators [11] constructed two families of mixed finite elements for second order elliptic problems, including the $\mathcal{H}(\mathbf{div})$-conforming triangular element. Further generalization to three dimensions had been carried out in 1987 by Brezzi, *et al.* [12]. From the perspective of differential forms Hipmair [13] gave a canonical construction of the $\mathcal{H}(\mathbf{curl})$- and $\mathcal{H}(\mathbf{div})$-conforming $\mathbb{R}^n$ simplicial elements. See also the related works [14–17] from such a perspective. In addition to other results Ainsworth and Coyle [18] constructed hierarchical bases of arbitrary order for $\mathcal{H}(\mathbf{div})$-conforming tetrahedral element. With polynomial approximation of an odd-numbered degree, the issue with enforcing conformity arising from a hierarchical basis [19, 20] had also been addressed in [18]. For the $\mathcal{H}(\mathbf{div})$-conforming simplicial elements and using techniques different from those in [18] Zaglmayr [21] gave two sets of hierarchical bases of arbitrary order.

It is well known in the finite element community that a hierarchical basis is more suitable than a nodal basis for the p- and hp-adaptivity [22,23]. However, a critical issue with a hierarchical basis is that with a high-order approximation the conditioning of the mass and stiffness matrices becomes to worsen to the extent of rendering the approximation results questionable and even meaningless. The conditioning issue has been realized and addressed by various researchers in different context for several conforming approximations, for examples, in [24–30]. In this study we continue our previous efforts [28–30] and concentrate on constructing well-conditioned hierarchical bases for $\mathcal{H}(\mathbf{div})$-conforming simplicial elements. The conditioning issue with $\mathcal{H}(\mathbf{div})$-conforming simplicial elements has specifically never been dealt with in the studies [7–9, 11–13, 18, 21]. Nevertheless, this does not necessarily mean that the matrix conditioning is not an issue with a hierarchical basis for $\mathcal{H}(\mathbf{div})$-conforming simplicial elements. On the contrary, we show in this study that such an issue does exist, and the mass and stiffness matrices may become badly ill-conditioned with a high-order approximation, and that this issue is more pronounced with the three-dimensional $\mathcal{H}(\mathbf{div})$-conforming tetrahedral elements.

Our new construction is based upon the works [18, 21] and is inspired by the research on orthogonal polynomials of several variables [31]. For the $\mathcal{H}(\mathbf{div})$-conforming tetrahedral elements and to achieve the goal of rendering the hierarchical basis well conditioned, our strategy is classifying shape functions into several groups, each of which is associated with a geometrical identity of the canonical reference tetrahedron [18], and making the shape functions orthonormal within each group with respect to the reference element. This is made possible by adroitly applying one fundamental result - Proposition 2.3.8