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A Multigrid Method for a Model of the Implicit Immersed Boundary Equations

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Abstract. Explicit time stepping schemes for the immersed boundary method require very small time steps in order to maintain stability. Solving the equations that arise from an implicit discretization is difficult. Recently, several different approaches have been proposed, but a complete understanding of this problem is still emerging. A multigrid method is developed and explored for solving the equations in an implicit-time discretization of a model of the immersed boundary equations. The model problem consists of a scalar Poisson equation with conformation-dependent singular forces on an immersed boundary. This model does not include the inertial terms or the incompressibility constraint. The method is more efficient than an explicit method, but the efficiency gain is limited. The multigrid method alone may not be an effective solver, but when used as a preconditioner for Krylov methods, the speed-up over the explicit-time method is substantial. For example, depending on the constitutive law for the boundary force, with a time step 100 times larger than the explicit method, the implicit method is about 15-100 times more efficient than the explicit method. A very attractive feature of this method is that the efficiency of the multigrid preconditioned Krylov solver is shown to be independent of the number of immersed boundary points.

AMS subject classifications: 65M55, 65F08, 76M20, 76D99

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1 Introduction

The immersed boundary (IB) method was developed by Peskin [18] to solve the coupled equations of motion of viscous fluid with an immersed elastic boundary. The method was

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developed to simulate blood flow in the heart, and it has since been applied to many different biofluid applications, and it is increasingly being used in other engineering problems [12]. The method involves two coordinate systems and two discrete grids. The fluid variables are represented in Eulerian coordinates which are discretized by a fixed, Cartesian grid. The immersed structures are represented in moving Lagrangian coordinates. The structures move at the local fluid velocity, interpolated from the Eulerian grid to the Lagrangian grid. The forces generated by the deformation of the structures are transferred to the Eulerian grid and appear as a forcing term in the momentum balance equation for the fluid.

Typical implementations of the IB method use a fractional stepping approach to solve the coupled fluid and boundary equations. The fluid velocity and pressure are updated for fixed boundary position, and then the boundary position is updated from the new velocity. Because the fluid and boundary are updated separately, one can use standard methods for solving for the fluid motion. One reason for the popularity of the IB method is that many different applications can be simulated with minor changes to existing codes. However, in many applications the elastic time scales are well below the physical time scales of interest, which means that the IB equations are very numerically stiff. When alternating between updating the fluid velocity and boundary position, this stiffness requires that the time step be very small in order to maintain stability.

Much effort has been devoted to both understanding and alleviating the severe time step restriction of IB methods [5,14,21]. Early attempts at implicit methods were not very efficient and thus not competitive with explicit methods [25], and some semi-implicit methods still presented significant time step restrictions [10,11]. Newren et al. [14] analyzed the origin of instability in semi-implicit methods using energy arguments, and they gave sufficient conditions for schemes to be unconditionally stable in the sense that the total energy is bounded regardless of the size of the time step. Recently a variety of stable semi-implicit methods have been developed [3,7,8,15], as well as several fully implicit methods [9,13]. Of course, these methods require more sophisticated algorithms in which the velocity and boundary position are solved for simultaneously. These recent methods are generally competitive in efficiency with explicit methods, and in some special cases they can be faster by factors of hundreds. It remains an open question as to whether there is a general, robust implicit method that is easy to use and more efficient than the explicit method for large classes of problems, or whether specialized methods will need to be developed for specific problems.

Many implicit methods reduce the full IB equations (fluid and boundary) to equations on only the boundary [2,3,13]. These methods achieve a substantial speed-up over explicit methods when there are relatively few immersed boundary points [3]. In addition, some methods require that the boundaries be smooth, closed curves [7,8]. Newren et al. [15] explored Kryolv methods for solving the linearized IB equations for different test problems. The relative efficiency of the implicit methods depended on the problem, and unpreconditioned Krylov methods were at least comparable in speed to explicit methods. These results suggest that with appropriate preconditioning, this approach