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Conforming Hierarchical Basis Functions

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Abstract. A unified process for the construction of hierarchical conforming bases on a range of element types is proposed based on an ab initio preservation of the underlying cohomology. This process supports not only the most common simplicial element types, as are now well known, but is generalized to squares, hexahedra, prisms and importantly pyramids. Whilst these latter cases have received (to varying degrees) attention in the literature, their foundation is less well developed than for the simplicial case. The generalization discussed in this paper is effected by recourse to basic ideas from algebraic topology (differential forms, homology, cohomology, etc) and as such extends the fundamental theoretical framework established by the work of Hiptmair [16–18] and Arnold et al. [4] for simplices. The process of forming hierarchical bases involves a recursive orthogonalization and it is shown that the resulting finite element mass, quasi-stiffness and composite matrices exhibit exponential or better growth in condition number.

AMS subject classifications: 65M60, 65M38, 41A10

Key words: Finite elements, vector basis functions, differential forms.

1 Introduction

High order curl and div conforming finite and boundary elements have become increasingly popular over recent years due to the demand for accurate and efficient solvers in, for, example, electromagnetics. Generally, these higher order schemes provide higher convergence rates and are correspondingly more economical than their lower order counterparts. For problems with singular solutions, however, the convergence rate is independent of the order of these bases if elements with uniform size and polynomial order are employed. This makes the use of adaptive (both in terms of element size (h) and order (p)) schemes highly attractive. This paper concerns the construction of hierarchical conforming basis functions, which are essential to the development of such adaptive strategies.

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High order conforming bases on simplices, squares and hexahedra were proposed as far back as 1980 [24], comprising unisolvent spaces together with degrees of freedom identified with submanifolds. Numerous variations on this theme followed, eg [10, 14]. In many ways these constructions were founded on specific realizations of Nédélec's original idea.

Following the early work of Bossavit [7, 8], algebraic topology (AT) (principally differential forms, homology and cohomology) provided the essential tools on which such conforming bases could be placed on a sound theoretical footing. Subsequently these techniques were further developed by Hiptmair [16–18], Bossavit [9] and notably Arnold et al. [4] where the theoretical foundations for spaces on simplices are fully developed for conforming spaces of arbitrary dimension. The construction of spaces on other element geometries (squares, hexahedra, prisms, pyramids) has received less attention. Although constructions exist, these rarely follow the AT framework, relying instead on Nédélec's original prescription. Exceptions to this are Bluck et al. [5] in 2D. Importantly, the desire for geometric flexibility in finite element methods leads to the need for schemes which employ both tetrahedra and hexahedra. This necessarily leads to a requirement for prismatic and pyramidal elements. Development of conforming bases on pyramids is rare and most notable in this context is the work of Gradinaru & Hipmair [12] and Graglia et al. [13]. Such functions are not wholly polynomial in nature and as a result are not directly amenable to construction via the Koszul operator employed by Arnold et al.

The development of hierarchical conforming bases is relatively recent, with much of the early work due to Webb [27], Andersen et al. [2, 3] in the context of curl-conforming simplicial finite elements. Hiptmair [16] briefly discusses the construction of hierarchical bases in terms of spaces of differential forms, again applied to simplices. Ilic & Notaros [19] develop curl-conforming hierarchical bases on curvilinear hexahedra, although poor conditioning is an issue. This issue of poor conditioning of the resulting matrix equations frequently arises in the development of hierarchical schemes. Indeed the bases developed by Webb are also very poorly conditioned. Techniques to address this have been the subject of much recent work, notably Ilic & Notaros [20], Ingelstrom [21, 22] for multilevel FE solvers using simplicial elements, Abdul-Rahman at al. [1] and Schöberl [25]. It has been shown by Graglia et al. [15] that hierarchical curl conforming bases can be classified as type A, B or C according to their approximation properties. Most recently Xin & Cai [28] (for triangles) and Xin et al. [29] (for tetrahedra) have investigated the dependence of condition number on element order for certain type A hierarchical curl-conforming basis functions. In the work presented here, we will develop a very general construction on a wide range of element types and for both 1 and 2-forms (curl and div conforming respectively). We will follow the Graglia classification (where appropriate), extending them to 2-form cases and perform a conditioning analysis analogous to [29] for all element types.

The contributions contained in this paper are as follows:

1. The construction of conforming spaces of differential forms valid for all element types (simplices, hexahedra, prisms and pyramids) and of all orders, based on an ab initio preservation of the underlying cohomology.