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A Parallel Second Order Cartesian Method for Elliptic Interface Problems

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Abstract. We present a parallel Cartesian method to solve elliptic problems with complex immersed interfaces. This method is based on a finite-difference scheme and is second-order accurate in the whole domain. The originality of the method lies in the use of additional unknowns located on the interface, allowing to express straightforwardly the interface transmission conditions. We describe the method and the details of its parallelization performed with the PETSc library. Then we present numerical validations in two dimensions, assorted with comparisons to other related methods, and a numerical study of the parallelized method.

AMS subject classifications: 65N06, 65N12, 65Y05

Key words: Elliptic interface problem, Cartesian method, second-order scheme, interface unknowns.

1 Introduction

In this paper we aim to solve on Cartesian grids with an order two accuracy the following problem:

$$\nabla.(k\nabla u)=f, \qquad \text{on } \Omega=\Omega_1\cup\Omega_2, \tag{1.1}$$

$$[\![u]\!]=\alpha, \qquad \text{on } \Sigma, \tag{1.2}$$

$$[\![k\frac{\partial u}{\partial n}]\!]=\beta, \qquad \text{on } \Sigma, \tag{1.3}$$

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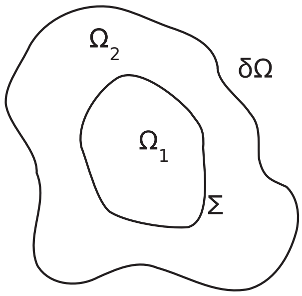

Figure 1: Geometry considered: two subdomains Ω_1 and Ω_2 separated by a complex interface Σ.

assorted with boundary conditions on $\delta\Omega$ defined as the boundary of Ω, and where $[\![\cdot]\!]$ means $\cdot_1 - \cdot_2$. As illustrated on Fig. 1, Ω consists in the union of two subdomains Ω_1 and Ω_2, separated by a complex interface Σ. This elliptic problem with discontinuities across an interface appears in numerous physical or biological models. Among the well-known applications are heat transfer, electrostatics, fluid dynamics, but similar elliptic problems arise for instance in tumor growth modelling, where one has to solve a pressure equation [11], or in the modelling of electric potential in biological cells [12]. In this latter case the jump of the solution across the interface is proportional to the interior normal derivative.

To solve an elliptic interface problem in the case of a complex interface, an alternative approach to body-fitted methods (see for instance [5, 10, 13]) is to discretize and solve the problem on a Cartesian grid. In this case, one takes into account the influence of the complex interface through modifications of the numerical scheme near the interface, without need of remeshing if the interface moves.

The first Cartesian grid method for elliptic problems was designed by Mayo in 1984 [30], and developed further in [31, 32]. In that work an integral equation was derived to solve elliptic interface problems with piecewise coefficients to second-order accuracy in maximum norm. Then LeVeque and Li (1994) [25] devised the very well known Immersed Interface Method (IIM). This method relies on Taylor expansions of the solution on each side of the interface, with a local coordinate transformation near the interface to express the jump conditions in an appropriate frame. The elliptic operator is discretized on each grid point near the interface with formulas accounting for the jumps across the interface. In order to find these formulas a linear system with six unknowns needs to be solved for each of the concerned grid points. The method is also second-order accurate in maximum norm. Numerous developments of the IIM have been performed. In the following lines we briefly evoke the most relevant. Li [26] developed a fast IIM algorithm for elliptic problems with piecewise constant coefficients. This version of IIM used auxiliary unknowns expressing the normal derivative at the interface. The fast IIM algorithm was generalized by Wiegmann and Bube in [42] under the name of Explicit Jump Immersed Interface Method (EJIIM). The EJIIM considers a classical finite-difference dis-