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A Compressed Sensing Approach for Partial Differential Equations with Random Input Data

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Abstract. In this paper, a novel approach for quantifying the parametric uncertainty associated with a stochastic problem output is presented. As with Monte-Carlo and stochastic collocation methods, only point-wise evaluations of the stochastic output response surface are required allowing the use of legacy deterministic codes and precluding the need for any dedicated stochastic code to solve the uncertain problem of interest. The new approach differs from these standard methods in that it is based on ideas directly linked to the recently developed compressed sensing theory. The technique allows the retrieval of the modes that contribute most significantly to the approximation of the solution using a minimal amount of information. The generation of this information, *via* many solver calls, is almost always the bottle-neck of an uncertainty quantification procedure. If the stochastic model output has a reasonably compressible representation in the retained approximation basis, the proposed method makes the best use of the available information and retrieves the dominant modes. Uncertainty quantification of the solution of both a 2-D and 8-D stochastic Shallow Water problem is used to demonstrate the significant performance improvement of the new method, requiring up to several orders of magnitude fewer solver calls than the usual sparse grid-based Polynomial Chaos (Smolyak scheme) to achieve comparable approximation accuracy.

AMS subject classifications: 60-08, 35J35, 35Q30

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1 Introduction

Uncertainty quantification has become a major concern for a wide range of communities. Indeed, in addition to providing accurate results, many simulation codes are now also expected to account for uncertainty in some of the intrinsic parameters of the problem and to provide confidence intervals and statistics of the outputs. Two basic types of uncertainty can be distinguished. Aleatory uncertainty may arise from the intrinsic variability of a physical quantity, *e.g.*, radioactive disintegration. The second type of uncertainty, referred to as the epistemic uncertainty, arises from a lack of knowledge of the considered quantity. In contrast to the aleatory uncertainty, the epistemic uncertainty may be reduced with additional knowledge on the quantity. The uncertain parameters may be initial or boundary conditions, geometric settings, constitutive material physical properties, etc., and their variability is suitably modeled using random variables. Specific methods must be used to infer the resulting uncertainty of the simulation outputs and provide statistical information such as mean, variance, quantiles, correlations, statistical moments or probability density functions of some quantities of interest, usually a functional of the simulation outputs. The probabilistic approach is a natural framework to achieve these objectives. While the original uncertain problem is sometimes of infinite dimension, reasonably accurate modeling often allows approximating the uncertainty sources with a finite set of real-valued random variables, for instance using a spectral decomposition technique, opening a route for a tractable computational solution method.

Indisputably, the most widely used approach to quantify the uncertainty associated with the solution of an uncertain problem is the Monte-Carlo approach. The probabilistic space is sampled and the associated deterministic problem is solved. From the collection of solutions arising from the N_{MC} samples, statistical information is derived. Several specific features explain the success of the Monte-Carlo approach. The main one is that the method relies only on the solution of deterministic problems, each solved for a given set of deterministic input parameters, avoiding the need for a dedicated uncertainty quantification-oriented code and allowing the use of legacy, well-validated and certified, deterministic codes that are used as a black-box. Further, the samples being drawn independently, it is embarrassingly straightforward to carry the N_{MC} simulations in parallel. The method is very general and robust and does not rely on assumptions on the solution. This robustness and simplicity come with a price that is most apparent in the poor $\mathcal{O}\left(N_{MC}^{-1/2}\right)$ convergence rate. Although numerous variants of the original Monte-Carlo method have been proposed, modifying the functional evaluated (Importance Sampling) or the way independent samples are generated (quasi-Monte-Carlo, Stratified Sampling, etc.), the convergence rate remains unchanged, with only the associated constant improved. This low convergence rate leads to requiring an unacceptably large number of simulations to compute reasonably converged statistics, precluding the use of Monte-Carlo methods in cases the deterministic simulation computational time is large. However, in contrast with other methods, the $\mathcal{O}\left(N_{MC}^{-1/2}\right)$ convergence rate is insensitive to the stochastic dimension of the uncertainty sources, making the Monte-Carlo