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Galerkin-Laguerre Spectral Solution of Self-Similar Boundary Layer Problems

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Abstract. In this work the Laguerre basis for the biharmonic equation introduced by Jie Shen is employed in the spectral solution of self-similar problems of the boundary layer theory. An original Petrov-Galerkin formulation of the Falkner-Skan equation is presented which is based on a judiciously chosen special basis function to capture the asymptotic behaviour of the unknown. A spectral method of remarkable simplicity is obtained for computing Falkner-Skan-Cooke boundary layer flows. The accuracy and efficiency of the Laguerre spectral approximation is illustrated by determining the linear stability of nonseparated and separated flows according to the Orr-Sommerfeld equation. The pentadiagonal matrices representing the derivative operators are explicitly provided in an Appendix to aid an immediate implementation of the spectral solution algorithms.

AMS subject classifications: 34L16, 34L30, 65L60, 76E05

Key words: Laguerre polynomials, semi-infinite interval, boundary layer theory, Falkner-Skan equation, Cooke equation, Orr-Sommerfeld equation, linear stability of parallel flows.

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1 Introduction

The interest in self-similar solutions of the boundary layer equation started in the first half of last century when, a few years after Prandtl formulated the boundary layer equations in 1904, Blasius obtained the ordinary differential equation for the self-similar boundary layer on a flat plate at zero incidence, see [12] and the references therein, and Falkner and Skan [9] derived the general equation for self-similar boundary layers, which is the subject of the present work. Despite such a long history, this problem still collects the interest of researchers and new numerical approaches are tested on it.

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Two main difficulties have to be faced in the numerical approximation of the Falkner-Skan equation. First, the interval where the solution is sought for is semi-infinite. Second, while conditions on both the unknown and its first derivative supplement the nonlinear differential equation at the wall, at infinity only the first derivative is assigned.

Starting from the early work of Hartree [13], so many different approaches have been proposed to tackle these difficulties that a full account of them would be the subject for a review article and out of the scope of the present paper, so we will provide just a few references. Here, it will suffice to say that two approaches have been mainly adopted to overcome the difficulty arising from the semi-infinite domain: truncation and mapping. After the problem has been reduced to a finite interval problem, the second difficulty can be addressed by solving directly the boundary value problem by means of finite differences [1], finite elements [2] or spectral methods [11]. Alternatively, exploiting shooting methods, the problem can be transformed into an initial value problem, see for instance [5]. A third approach is to restate the problem as a free boundary problem, as first attempted by R. Fazio [10] in the context of self-similar boundary layers and, more recently, by Zhang and Chen [24].

Problems on a semi-infinite domain, however, can be solved without any truncation or rescaling if one introduces Laguerre polynomials/functions [22, 23] or rational functions [4] to build a suitable basis. Recently, Parand and coworkers [17,18] proposed spectral collocation solvers for the Blasius equation which employ Laguerre functions and rational Chebyshev functions, respectively. With respect to truncation methods, these approaches do not require to properly choose the domain size, but allow the user to select spatial resolution of the basis by appropriately selecting a coordinate scaling factor.

While being quite convenient and simple to implement, the collocation approach in spectral methods leads to discrete operators which are full, nonsymmetric, even for self-adjoint problems, and poorly conditioned, especially when high order derivatives are present. On the contrary, in a series of works spanning different coordinate systems and domain geometries, Jie Shen has shown that the spectral Galerkin method can lead to very simple, sparse and optimally conditioned discrete operators. Moreover, the Galerkin method guarantees optimal error estimates. These good properties apply, in particular, to spectral methods on the semi-infinite interval based on Laguerre functions, as shown in [22]. In this work we adopt two different Laguerre spectral bases, both proposed in [22], to solve the Falkner-Skan-Cooke equations and to implement a linear stability solver for both 2D and 3D problems in the parallel flow approximation. By virtue of the adopted bases, boundary conditions are taken into account very easily. Moreover, when discretized, the derivative operators with constant coefficients lead to sparse matrices whose entries have been computed in closed form.

Unfortunately, the Galerkin method can not be applied straightforwardly to solve neither the Blasius nor the Falkner-Skan equation, since Laguerre functions do not satisfy the asymptotic behaviour of the solution, whose limit at infinity is unknown a-priori. To overcome this difficulty, we propose to modify the standard Galerkin method for solutions in L^2 to a Petrov-Galerkin method by resorting to a change of variable and in-