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A Novel Numerical Method of $\mathcal{O}(h^4)$ for Three-Dimensional Non-Linear Triharmonic Equations

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Abstract. In this article, we present two new novel finite difference approximations of order two and four, respectively, for the three dimensional non-linear triharmonic partial differential equations on a compact stencil where the values of u, $\partial^2 u/\partial n^2$ and $\partial^4 u/\partial n^4$ are prescribed on the boundary. We introduce new ideas to handle the boundary conditions and there is no need to discretize the derivative boundary conditions. We require only 7- and 19-grid points on the compact cell for the second and fourth order approximation, respectively. The Laplacian and the biharmonic of the solution are obtained as by-product of the methods. We require only system of three equations to obtain the solution. Numerical results are provided to illustrate the usefulness of the proposed methods.

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1 Introduction

We are concerned with the numerical solution of three dimensional non-linear triharmonic partial differential equation of the form:

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$$\begin{aligned}&\nabla^6 u(x,y,z)\\ \equiv&\frac{\partial^6 u}{\partial x^6}+\frac{\partial^6 u}{\partial y^6}+\frac{\partial^6 u}{\partial z^6}+6\frac{\partial^6 u}{\partial x^2\partial y^2\partial z^2}+3\Big(\frac{\partial^6 u}{\partial x^4\partial y^2}+\frac{\partial^6 u}{\partial x^2\partial y^4}\\ &\quad+\frac{\partial^6 u}{\partial z^4\partial y^2}+\frac{\partial^6 u}{\partial z^2\partial y^4}+\frac{\partial^6 u}{\partial x^4\partial z^2}+\frac{\partial^6 u}{\partial x^2\partial z^4}\Big)\\ =&f(x,y,z,u,u_x,u_y,u_z,\nabla^2 u,\nabla^2 u_x,\nabla^2 u_y,\nabla^2 u_z,\nabla^4 u,\nabla^4 u_x,\nabla^4 u_y,\nabla^4 u_z),\quad (x,y,z)\in\Omega, \end{aligned}\tag{1.1}$$

where $\Omega=\{(x,y,z)|0<x,y,z<1\}$ is the solution region with boundary $\partial\Omega$ and

$$\nabla^2 u\equiv\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\quad\text{and}\quad \nabla^4 u\equiv\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}+\frac{\partial^4 u}{\partial z^4}+2\Big(\frac{\partial^4 u}{\partial x^2\partial y^2}+\frac{\partial^4 u}{\partial z^2\partial y^2}+\frac{\partial^4 u}{\partial x^2\partial z^2}\Big)$$

represent the three dimensional Laplacian and biharmonic of the function $u(x,y,z)$. We assume that the solution $u(x,y,z)$ is smooth enough to maintain the order of accuracy as high as possible of the finite difference schemes under consideration.

The values of

$$u,\quad \frac{\partial^2 u}{\partial n^2}\quad\text{and}\quad\frac{\partial^4 u}{\partial n^4}\ \text{are prescribed on the boundary } \partial\Omega. \tag{1.2}$$

The boundary conditions prescribed by (1.2) are called second kind boundary conditions. Since the grid lines are parallel to coordinate axes, we assume that the boundary values prescribed by (1.2) must satisfy the consistency conditions at all twelve edges and eight corner points of the boundary $\partial\Omega$, i.e., $u_{xx}=u_{yy}=u_{zz}$ and $u_{xxxx}=u_{yyyy}=u_{zzzz}$ at all corner points, $u_{xx}=u_{yy}$ and $u_{xxxx}=u_{yyyy}$ at all points on the lines parallel to z-axis, $\cdots$, etc. The triharmonic equation is a sixth order elliptic partial differential equation which is encountered in viscous flow problems. Not many researchers have tried to solve the triharmonic equations, because it is difficult to discretize the differential equations and boundary conditions on a compact cell and moreover triharmonic problems require large computing power and place huge amount of memory requirements on the computational systems. Such computing power has only recently begun to become available for academic research. Different techniques for the numerical solution of the 2D non-linear biharmonic and 3D non-linear biharmonic equations have been considered in the literature, but not for the 3D non-linear triharmonic equations. A popular technique is to split the biharmonic equation into two coupled Poisson equations each of which may be discretized using the standard approximations and solving using any of the Poisson solvers. Difficulty with this approach is that the boundary conditions for the new variable are undefined and need to be approximated at the boundary. Smith [1] and Ehrlich [2,3] have solved 2D biharmonic equations using coupled second order accurate finite difference approximations. Bauer and Riess [4] have used block iterative method to solve the equation. Later, Kwon et al. [5], Stephenson [6], Evans and Mohanty [7], Mohanty et