

A Distributed Control Approach for the Boundary Optimal Control of the Steady MHD Equations

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Abstract. A new approach is presented for the boundary optimal control of the MHD equations in which the boundary control problem is transformed into an extended distributed control problem. This can be achieved by considering boundary controls in the form of lifting functions which extend from the boundary into the interior. The optimal solution is then sought by exploring all possible extended functions. This approach gives robustness to the boundary control algorithm which can be solved by standard distributed control techniques over the interior of the domain.

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1 Introduction

Fluid flows interacting with magnetic fields is a feature in many science and engineering settings such as fusion technology, fission nuclear reactors cooled using liquid metals, and submarine propulsion devices [5, 18]. Such flows are described by the magnetohydrodynamic (MHD) system of equations for which numerous formulations have been proposed and analyzed in the literature, based on differing physical assumptions about the MHD model; see, e.g., [9, 15, 20, 21, 23]. For example, for the description of electromagnetic phenomena, the Maxwell equations or some related simplifications employing different sets of state variables have been used, with the state variables consisting of one combination or another of quantities such as the magnetic field, the current density, the

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electric field, and the electric potential [9]. The mechanical behavior of the fluid flow is often described by the Navier-Stokes equations. It is well known that, whereas the Navier-Stokes equations are posed over the region occupied by the fluid, the Maxwell equations extend to all of three-dimensional space [20]. In addition, initial, boundary, and interface conditions are imposed; the specific choices for these constraints help define specific physical models and affect aspects of the mathematical models such as weak formulations along with the choice of the associated function spaces.

Several approaches have been proposed for optimal control problems constrained by the MHD equations; see, e.g., [9, 12, 17]. Compared to the case of distributed controls, standard approaches for treating boundary control problems are not entirely straightforward to implement numerically. In fact, boundary controls involve normal or tangential components of the magnetic field so that the direct implementation of such controls causes substantial difficulties on general domains. Furthermore, such implementations often lead to unnecessarily smooth controls [11, 20] or involve overpenalization that can adversely affect the accuracy of approximations and the conditioning of discretized systems.

We instead introduce a novel approach in which the boundary control problem is transformed, through lifting functions, into a distributed control problem from which an *optimal distributed magnetic field control* may be determined. By appropriately restricting the optimal distributed control to the boundary, any and all possible boundary controls can be determined.

The paper is organized as follows. In Section 2, we introduce the MHD optimal boundary control problem. We also introduce our modification to that problem that allows us to instead determine an optimal distributed control from which the boundary control may be defined by restriction. Then, in Section 3, we provide a weak formulation of the MHD state equations suitable for our purposes and prove the existence of a solution of those equations. This result is needed to obtain the results of Section 4 in which a precise, functional analytic definition of the optimal control is given, followed by a proof of the existence of an optimal solution. The technique of Lagrange multipliers, the first-order necessary condition, and the optimality system are discussed in Sections 4.2 and 4.3. Section 5 contains the results of some numerical experiments.

2 Description of the optimal control problem

The optimal control problem we consider consists of a cost or objective functional, a set of control functions, and a set of state equations that act as constraints.

Let $\Omega \subset \mathbb{R}^3$ denote an open bounded connected domain with $C^{1,1}$ boundary Γ . We denote by Γ_1 a subset of Γ with positive surface measure. For the constraint or state