

## Numerical Solutions of Coupled Nonlinear Schrödinger Equations by Orthogonal Spline Collocation Method

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**Abstract.** In this paper, we present the use of the orthogonal spline collocation method for the semi-discretization scheme of the one-dimensional coupled nonlinear Schrödinger equations. This method uses the Hermite basis functions, by which physical quantities are approximated with their values and derivatives associated with Gaussian points. The convergence rate with order  $\mathcal{O}(h^4 + \tau^2)$  and the stability of the scheme are proved. Conservation properties are shown in both theory and practice. Extensive numerical experiments are presented to validate the numerical study under consideration.

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**Key words:** Coupled nonlinear Schrödinger equations, orthogonal spline collocation method, conservation law.

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## 1 Introduction

The coupled nonlinear Schrödinger (CNLS) equations were first derived 30 years ago by Benney and Newell [6] for two interacting nonlinear packets in dispersive and conservative systems. Since then, the CNLS equations have been appeared in a great variety of physical situations. Its applications can be found in many areas of physics, including nonlinear optics and plasma physics, see, e.g., [1, 16, 22, 24, 28]. These equations also

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model a beam propagation inside crystals or photorefractives as well as water wave interactions. In this paper, we consider the following CNLS equations:

$$i \frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2} + (|\varphi|^2 + \beta |\psi|^2) \varphi = 0, \quad (1.1a)$$

$$i \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + (|\psi|^2 + \beta |\varphi|^2) \psi = 0, \quad (1.1b)$$

where  $\varphi$  and  $\psi$  represent complex amplitudes of two polarization components,  $\beta$  is a real-valued cross-phase modulation coefficient,  $i = \sqrt{-1}$ ,  $x$  is the space variable and  $t$  is the time variable. If  $\beta = 0$ , Eq. (1.1) becomes two copies of a single nonlinear Schrödinger equation which is integrable; when  $\beta = 1$ , Eq. (1.1) is known as the Manakov system which is also integrable. In all the other cases, the situations are much more complicated from different viewpoints. These equations have been studied intensively in the past 10 years [8, 14, 18]. Much work has been done on collisions in a large array of physical systems. Various collision scenarios, such as transmission, reflection, annihilation, trapping, creation of solitary waves and even mutual spiraling, have been reported.

There are a great deal of numerical methods used to solve the CNLS equations. Antoine et al. [2] give a review to discuss different techniques to solve numerically the time-dependent Schrödinger equation on unbounded domains. Klein et al. [15] propose a hierarchy of novel absorbing boundary conditions for the one-dimensional stationary Schrödinger equation with general potential. The time-splitting spectral method for solving a general model of wave optical interactions is obtained by Bao et al. [3–5, 30]. Xu and Shu [25–27] develop local discontinuous Galerkin methods for solving high-order time-dependent partial differential equations including CNLS equations. Ismail and Taha [14] introduce a finite difference method for the numerical simulation of the CNLS equations. The multi-symplectic splitting method is proposed to solve the CNLS equations in [9] and the constrained interpolation profile-basis set method is considered in [21]. Wang [23] presents a numerical solution of the single and coupled nonlinear Schrödinger equations using a split-step finite difference method. However, in these numerical simulations and computations, many constraints are required in order to keep the accuracy and stability. Moreover, many properties of the system, such as energy conservation and momentum conservation, are neglected.

The purpose of this paper is to investigate the use of the orthogonal spline collocation (OSC) method with the piecewise Hermite cubic polynomials for the spatial discretization of Eq. (1.1). This method has evolved as a valuable technique for the solution of many types of partial differential equations; see [11] for a comprehensive survey. The popularity of such a method is in part to its conceptual simplicity and ease of implementation. Another attractive feature of the OSC method is their superconvergence. One obvious advantage of the OSC method over the finite element method is that the calculation of the coefficient matrices is very efficient since no integral calculation is required. Another advantage of this method is that it systematically incorporates boundary conditions and interface conditions. In comparison with finite difference methods, spline