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A Reconstructed Discontinuous Galerkin Method for the Euler Equations on Arbitrary Grids

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Abstract. A reconstruction-based discontinuous Galerkin (RDG(P1P2)) method, a variant of P1P2 method, is presented for the solution of the compressible Euler equations on arbitrary grids. In this method, an in-cell reconstruction, designed to enhance the accuracy of the discontinuous Galerkin method, is used to obtain a quadratic polynomial solution (P2) from the underlying linear polynomial (P1) discontinuous Galerkin solution using a least-squares method. The stencils used in the reconstruction involve only the von Neumann neighborhood (face-neighboring cells) and are compact and consistent with the underlying DG method. The developed RDG method is used to compute a variety of flow problems on arbitrary meshes to demonstrate its accuracy, efficiency, robustness, and versatility. The numerical results indicate that this RDG(P1P2) method is third-order accurate, and outperforms the third-order DG method (DG(P2)) in terms of both computing costs and storage requirements.

AMS subject classifications: 65M60, 65M99, 76M25, 76M10

Key words: Discontinuous Galerkin methods, least-squares reconstruction, compressible Euler equations.

1 Introduction

The discontinuous Galerkin methods [1–28] (DGM) have recently become popular for the solution of systems of conservation laws. Originally introduced for the solution of neutron transport equations [1], nowadays they are widely used in computational fluid dynamics, computational acoustics, and computational magneto-hydrodynamics. The discontinuous Galerkin methods combine two advantageous features commonly associated

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with finite element and finite volume methods. As in classical finite element methods, accuracy is obtained by means of high-order polynomial approximation within an element rather than by wide stencils as in the case of finite volume methods. The physics of wave propagation is, however, accounted for by solving the Riemann problems that arise from the discontinuous representation of the solution at element interfaces. In this respect, the DG methods are similar to finite volume methods. The discontinuous Galerkin methods have many attractive features: 1) They have several useful mathematical properties with respect to conservation, stability, and convergence; 2) The methods can be easily extended to higher-order ($>2^{\mathrm{nd}}$) approximation; 3) The methods are well suited for complex geometries since they can be applied on unstructured grids. In addition, the methods can also handle non-conforming elements, where the grids are allowed to have hanging nodes; 4) The methods are highly parallelizable, as they are compact and each element is independent. Since the elements are discontinuous, and the inter-element communications are minimal, domain decomposition can be efficiently employed. The compactness also allows for structured and simplified coding for the methods; 5) They can easily handle adaptive strategies, since refining or coarsening a grid can be achieved without considering the continuity restriction commonly associated with the conforming elements. The methods allow easy implementation of hp-refinement, for example, the order of accuracy, or shape, can vary from element to element; 6) They have the ability to compute low Mach number flow problems without recourse to the time-preconditioning techniques normally required for the finite volume methods. In contrast to the enormous advances in the theoretical and numerical analysis of the DGM, the development of a viable, attractive, competitive, and ultimately superior DG method over the more mature and well-established second order methods is relatively an untouched area. This is mainly due to the fact that the DGM have a number of weaknesses that have yet to be addressed, before they can be robustly used for flow problems of practical interest in a complex configuration environment. In particular, there are three most challenging and unresolved issues in the DGM: a) how to efficiently discretize diffusion terms required for the Navier-Stokes equations, b) how to effectively control spurious oscillations in the presence of strong discontinuities, and c) how to develop efficient time integration schemes for time accurate and steady-state solutions. Indeed, compared to the finite element methods and finite volume methods, the DG methods require solutions of systems of equations with more unknowns for the same grids. Consequently, these methods have been recognized as expensive in terms of both computational costs and storage requirements.

DG methods are indeed a natural choice for the solution of the hyperbolic equations, such as the compressible Euler equations. However, the DG formulation is far less certain and advantageous for the compressible Navier-Stokes equations, where viscous and heat fluxes exist. A severe difficulty raised by the application of the DG methods to the Navier-Stokes equations is the approximation of the numerical fluxes for the viscous fluxes, that has to properly resolve the discontinuities at the interfaces. Taking a simple arithmetic mean of the solution derivatives from the left and right is inconsistent, be-