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Numerical Simulation of a Weakly Nonlinear Model for Internal Waves

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Abstract. Internal waves arise in a wide array of oceanographic problems of both theoretical and engineering interest. In this contribution we present a new model, valid in the weakly nonlinear regime, for the propagation of disturbances along the interface between two ideal fluid layers of infinite extent and different densities. Additionally, we present a novel high-order/spectral algorithm for its accurate and stable simulation. Numerical validation results and simulations of wave-packet evolution are provided.

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Key words: Internal waves, weakly nonlinear waves, high-order spectral methods, Dirichlet Neumann operators, surface formulation.

1 Introduction

An internal water-wave propagates along an interface between two fluids with varying densities due to variations in temperature and salinity [18]. Internal waves are ubiquitous in the world's oceans and, as seen by satellite [15], they have been observed to travel for hundreds of miles within the oceans. Internal waves are important both as a significant source of energy and momentum transportation and in their interactions with ocean dynamics and topographies. In addition to their many engineering and other practical applications, they inspire many difficult questions in both theoretical and numerical analysis.

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The classical formulation of the internal water-wave problem uses Euler's equations with kinematic and dynamic boundary conditions at the free interface. Well-known approximation models include, most notably, the Korteweg-de Vries, Benjamin-Ono, Boussinesq, Nonlinear Schrödinger and Intermediate Long Wave equations [4, 25]. Alternatively, one could reformulate the classical system as a boundary integral equation, a Hamiltonian system, or as evolution equations involving operators (see, e.g., [7, 13] for recent examples).

This paper is a numerical study of time-dependent internal wave propagation and for this we focus on two-fluid systems which Koop & Butler [16] believe are sufficient to capture the physics of the problem. For simplicity, we neglect forces other than gravity and assume infinite extent in both the upper and lower layers, but we note that our method can easily be extended to include additional forces (e.g., surface tension), finite layers and rigid-lid assumptions. The model derivation is based on a perturbative approach [26] in the weakly nonlinear regime (see also [5, 25]) and our evolution equations govern interface variables involving interface integral operators, e.g., Dirichlet-Neumann operators (DNOs). The interface variables, first introduced by Benjamin & Bridges [1, 2] and Craig & Groves [6], are analogous to the surface variables identified by Zakharov [33] for a Hamiltonian formulation of the classical single-fluid water-wave problem. The DNO, a linear operator that maps Dirichlet boundary data to Neumann boundary data [27] was introduced to the water-wave problem by Craig and Sulem [10] (see also the work of Milder [20, 21] and Milder & Sharp [22, 23] in the setting of electromagnetic scattering) to make Zakharov's formulation more explicit. The interface formulation is appealing for a number of reasons, not the least of which is that it reduces the problem dimension. In [10] Craig and Sulem used the new formulation to numerically simulate surface gravity waves. Our work is motivated by [6, 10] and can be considered both an extension of their work to the two-fluid case and a complement to the research of [7, 11] who also use this formulation for such simulations.

To the authors' knowledge, this work presents, for the first time, numerical simulations of truncated Dirichlet-Neumann operators which have not appeared in the literature before. We also point out that due to the interface formulation and the spectrally accurate nature of our algorithm, our approach *cannot* be surpassed in terms of accuracy and speed (though integral equation approaches will have roughly the same operation counts).

Given our model equations, the evolution of the internal wave is simulated by a Fourier collocation spectral method [12] in the spatial variable and a fourth-order Runge-Kutta algorithm [3] for time-stepping. The main effort lies with the approximation of the operators and once this is accomplished, the numerical method is fast, highly accurate and easy to implement. The full literature of numerical schemes to simulate the motion of a free surface or interface in the Euler equations of ideal fluid motion is far too vast to recapitulate here. For the interested reader we recommend the survey articles of [14, 19, 30–32].

This paper is organized as follows: In Section 2.1 we outline the classical formulation