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A Finite Volume Upwind-Biased Centred Scheme for Hyperbolic Systems of Conservation Laws: Application to Shallow Water Equations

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Abstract. We construct a new first-order central-upwind numerical method for solving systems of hyperbolic equations in conservative form. It applies in multidimensional structured and unstructured meshes. The proposed method is an extension of the UFORCE method developed by Stecca, Siviglia and Toro [25], in which the upwind bias for the modification of the staggered mesh is evaluated taking into account the smallest and largest wave of the entire Riemann fan. The proposed first-order method is shown to be identical to the Godunov upwind method in applications to a 2×2 linear hyperbolic system. The method is then extended to non-linear systems and its performance is assessed by solving the two-dimensional inviscid shallow water equations. Extension to second-order accuracy is carried out using an ADER-WENO approach in the finite volume framework on unstructured meshes. Finally, numerical comparison with current competing numerical methods enables us to identify the salient features of the proposed method.

AMS subject classifications: 65M08, 76M12

Key words: Conservative hyperbolic systems, centred schemes, unstructured meshes, numerical fluxes, shallow water equations, FORCE, upwind-biased.

1 Introduction

1.1 Preliminaries

We consider a general system of non-linear conservation laws in α space dimensions:

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$$\partial_t \mathbf{Q} + \text{div}\left(\underline{\underline{\mathbf{F}}}(\mathbf{Q})\right) = \mathbf{0}, \tag{1.1}$$

where $\underline{\underline{\mathbf{F}}}(\mathbf{Q})$ is the flux tensor.

We assume a conforming tessellation $\mathcal{T}_\Omega$ of the computational domain $\Omega \subset \mathbb{R}^\alpha$ by n_e elements T_i such that:

$$\mathcal{T}_\Omega = \bigcup_{i=1}^{n_e} T_i. \tag{1.2}$$

Each element T_i has n_f plane interfaces S_j of size $|S_j|$, with associated outward pointing face normal vectors $\vec{n}_j$. Element T_i, having size $|T_i|$, is sub-divided into subvolumes V_j^- generated by connecting the barycentre of T_i with the vertices of S_j. The corresponding adjacent subvolume in the neighbouring element that shares face S_j with element T_i is denoted as V_j^+. Fig. 1 illustrates the above definitions and notation for the two-dimensional case. Note that the intersection of V_j^- and V_j^+ gives the interface S_j of the element T_i. With reference to Fig. 1 we distinguish two kinds of elements: *primary elements* T_i, at which the solution is sought at each time step, and *secondary elements* formed by $V_j^- \bigcup V_j^+$, for $j = 1,2,3$.

Finite volume schemes are obtained by integration of the conservation law (1.1) over a space-time control volume $T_i \times [t^n, t^{n+1}]$, yielding:

$$\mathbf{Q}_i^{n+1} = \mathbf{Q}_i^n - \frac{\Delta t}{|T_i|} \sum_{j=1}^{n_f} \int_{S_j} \underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}} \left(\mathbf{Q}_i^n, \mathbf{Q}_j^n\right) \cdot \vec{n}_j \mathrm{d}\vec{x}, \tag{1.3}$$

where $\mathbf{Q}_i^n$ is the cell average at time level n and $\Delta t = t^{n+1} - t^n$ is the time step. Two different approaches are available for determining $\underline{\underline{\mathbf{F}}}_{j+\frac{1}{2}}$. The first approach is the *upwind approach*, represented by Godunov's method [9] and the second is the *centred approach*, typically represented by the Lax-Friedrichs flux and variations of it [18]. For a comprehensive presentation of upwind, and also some centred methods, see for example [27] and references therein.

In this paper we derive a central-upwind method which partially uses upwind information, while retaining the simplicity and efficiency of a centred scheme. Kurganov and Tadmor put forward an analogous idea in their *central-upwind* approach [17], using an adaptive staggered mesh. Their scheme is based on a modification of the centred scheme of Nessyahu and Tadmor [21], where the staggered mesh is fixed. Extensions to multidimensions of the scheme of Nessyahu and Tadmor [21] has been obtained by Jiang and Tadmor [13] and by Arminjon and collaborators [1]. Multi-dimensional extensions of the scheme of Kurganov and Tadmor have been presented in [14] (Cartesian version) and [16] (unstructured version), while a modified version of the scheme optimised for treating contact discontinuities, which makes use of partial characteristic decomposition, has been presented in [15].