A Finite Volume Upwind-Biased Centred Scheme for Hyperbolic Systems of Conservation Laws: Application to Shallow Water Equations

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Received 18 May 2011; Accepted (in revised version) 7 December 2011

Communicated by Song Jiang

Available online 17 April 2012

Abstract. We construct a new first-order central-upwind numerical method for solving systems of hyperbolic equations in conservative form. It applies in multidimensional structured and unstructured meshes. The proposed method is an extension of the UFORCE method developed by Stecca, Siviglia and Toro [25], in which the upwind bias for the modification of the staggered mesh is evaluated taking into account the smallest and largest wave of the entire Riemann fan. The proposed first-order method is shown to be identical to the Godunov upwind method in applications to a 2 × 2 linear hyperbolic system. The method is then extended to non-linear systems and its performance is assessed by solving the two-dimensional inviscid shallow water equations. Extension to second-order accuracy is carried out using an ADER-WENO approach in the finite volume framework on unstructured meshes. Finally, numerical comparison with current competing numerical methods enables us to identify the salient features of the proposed method.

AMS subject classifications: 65M08, 76M12

Key words: Conservative hyperbolic systems, centred schemes, unstructured meshes, numerical fluxes, shallow water equations, FORCE, upwind-biased.

1 Introduction

1.1 Preliminaries

We consider a general system of non-linear conservation laws in a space dimensions:

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∂tQ + \text{div}(F(Q)) = 0, \quad (1.1)

where $F(Q)$ is the flux tensor.

We assume a conforming tessellation $\mathcal{T}_\Omega$ of the computational domain $\Omega \subset \mathbb{R}^n$ by $n_e$ elements $T_i$ such that:

$$\mathcal{T}_\Omega = \bigcup_{i=1}^{n_e} T_i. \quad (1.2)$$

Each element $T_i$ has $n_f$ plane interfaces $S_j$ of size $|S_j|$, with associated outward pointing face normal vectors $\vec{n}_j$. Element $T_i$, having size $|T_i|$, is sub-divided into subvolumes $V^-_j$ generated by connecting the barycentre of $T_i$ with the vertices of $S_j$. The corresponding adjacent subvolume in the neighbouring element that shares face $S_j$ with element $T_i$ is denoted as $V^+_j$. Fig. 1 illustrates the above definitions and notation for the two-dimensional case. Note that the intersection of $V^-_j$ and $V^+_j$ gives the interface $S_j$ of the element $T_i$. With reference to Fig. 1 we distinguish two kinds of elements: primary elements $T_i$, at which the solution is sought at each time step, and secondary elements formed by $V^-_j \cup V^+_j$, for $j = 1, 2, 3$.

Finite volume schemes are obtained by integration of the conservation law (1.1) over a space-time control volume $T_i \times [t^n, t^{n+1}]$, yielding:

$$Q^{n+1}_i = Q^n_i - \frac{\Delta t}{|T_i|} \sum_{j=1}^{n_f} \int_{S_j} F_j \left( \frac{Q^n_i + Q^n_j}{2} \right) \cdot \vec{n}_j \, d\vec{x}, \quad (1.3)$$

where $Q^n_i$ is the cell average at time level $n$ and $\Delta t = t^{n+1} - t^n$ is the time step. Two different approaches are available for determining $F_j$. The first approach is the upwind approach, represented by Godunov’s method [9] and the second is the centred approach, typically represented by the Lax-Friedrichs flux and variations of it [18]. For a comprehensive presentation of upwind, and also some centred methods, see for example [27] and references therein.

In this paper we derive a central-upwind method which partially uses upwind information, while retaining the simplicity and efficiency of a centred scheme. Kurganov and Tadmor put forward an analogous idea in their central-upwind approach [17], using an adaptive staggered mesh. Their scheme is based on a modification of the centred scheme of Nessyahu and Tadmor [21], where the staggered mesh is fixed. Extensions to multidimensions of the scheme of Nessyahu and Tadmor [21] has been obtained by Jiang and Tadmor [13] and by Arminjon and collaborators [1]. Multi-dimensional extensions of the scheme of Kurganov and Tadmor have been presented in [14] (Cartesian version) and [16] (unstructured version), while a modified version of the scheme optimised for treating contact discontinuities, which makes use of partial characteristic decomposition, has been presented in [15].