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A Numerical Scheme for the Quantum Fokker-Planck-Landau Equation Efficient in the Fluid Regime

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Abstract. We construct an efficient numerical scheme for the quantum Fokker-Planck-Landau (FPL) equation that works uniformly from kinetic to fluid regimes. Such a scheme inevitably needs an implicit discretization of the nonlinear collision operator, which is difficult to invert. Inspired by work [9] we seek a linear operator to penalize the quantum FPL collision term $\mathcal{Q}_{qFPL}$ in order to remove the stiffness induced by the small Knudsen number. However, there is no suitable simple quantum operator serving the purpose and for this kind of operators one has to solve the complicated quantum Maxwellians (Bose-Einstein or Fermi-Dirac distribution). In this paper, we propose to penalize $\mathcal{Q}_{qFPL}$ by the "classical" linear Fokker-Planck operator. It is based on the observation that the classical Maxwellian, with the temperature replaced by the internal energy, has the same first five moments as the quantum Maxwellian. Numerical results for Bose and Fermi gases are presented to illustrate the efficiency of the scheme in both fluid and kinetic regimes.

AMS subject classifications: 35Q84, 65L04, 76Y05

Key words: Quantum Fokker-Planck-Landau equation, fluid limit, asymptotic-preserving scheme.

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1 Introduction

The Fokker-Planck-Landau (FPL) equation is a kinetic model widely used in plasma physics. It describes the time evolution of charged particles in a plasma [21, 22]. When the quantum effects of particles are taken into account, for example, several bosons can occupy the same quantum state while only one fermion can occupy a particular quantum state, one has to use the following so-called quantum Fokker-Planck-Landau equation,

$$\frac{\partial f}{\partial t}+v\cdot\nabla_x f=\frac{1}{\varepsilon}\mathcal{Q}_{qFPL}(f),\qquad x\in\Omega\subset\mathbb{R}^{d_x},\quad v\in\mathbb{R}^{d_v}, \tag{1.1}$$

where $f(t,x,v)\geq 0$ is the phase space distribution function depending on time t, position x and particle velocity v. ε is the Knudsen number which measures the degree of rarefiedness of the particles. It is the ratio of the mean free path and the typical length scale. The quantum collision operator $\mathcal{Q}_{qFPL}$ is given by

$$\mathcal{Q}_{qFPL}(f)(v)=\nabla_v\cdot\int_{\mathbb{R}^{d_v}}A(v-v_*)\left[f_*(1\pm\theta_0 f_*)\nabla_v f-f(1\pm\theta_0 f)\nabla_{v_*}f_*\right]dv_* \tag{1.2}$$

with $f=f(t,x,v)$ and $f_*=f(t,x,v_*)$. $A(z)=\Psi(|z|)\Pi(z)$ is a $d_v\times d_v$ semi-positive definite matrix and $\Pi(z)$ is the orthogonal projection onto the space orthogonal to z,

$$\Pi(z)=I-\frac{z\otimes z}{|z|^2},\quad I \text{ is the identity matrix.} \tag{1.3}$$

For inverse-power law interactions, $\Psi(|z|)=|z|^{\gamma+2}$ with $-3\leq\gamma\leq 1$. The case $\gamma=-3$ refers to the Coulomb potential which is of primary importance in plasma applications. The parameter $\theta_0=\hbar^{d_v}$, where $\hbar$ is the rescaled Planck constant. Here in (1.2) and the sequel, the upper sign will always correspond to the Bose gas (composed of bosons) while the lower sign to the Fermi gas (composed of fermions). For the latter f must also satisfy $f\leq 1/\theta_0$ by the Pauli exclusion principle.

Unlike the classical FPL equation, very few studies have been conducted on the quantum FPL equation. See [7] for a formal derivation from the quantum Boltzmann equation in the grazing collision limit and [23] for a spectral analysis of its linearization near the equilibrium. In the spatially homogeneous setting, the well-posedness and regularity of the solution were established in [1,5] for Fermi-Dirac particles and the equilibrium states were rigorously determined in [2].

It is well-known that the equilibrium, in this context the quantum Maxwellian $\mathcal{M}_q$ (Bose-Einstein or Fermi-Dirac distribution), is reached when the Knudsen number ε goes to zero. Then we could instead consider the limiting hydrodynamic equations satisfied by the moments of $\mathcal{M}_q$. However, fluid equations are not adequate for many applications. Very often one has to deal with multiscale phenomena, where the Knudsen number varies between different regimes. Our goal in this paper is to design an efficient numerical scheme for the quantum FPL equation (1.1) that works uniformly for both kinetic and