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An Iterative Two-Fluid Pressure Solver Based on the Immersed Interface Method

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Abstract. An iterative solver based on the immersed interface method is proposed to solve the pressure in a two-fluid flow on a Cartesian grid with second-order accuracy in the infinity norm. The iteration is constructed by introducing an unsteady term in the pressure Poisson equation. In each iteration step, a Helmholtz equation is solved on the Cartesian grid using FFT. The combination of the iteration and the immersed interface method enables the solver to handle various jump conditions across two-fluid interfaces. This solver can also be used to solve Poisson equations on irregular domains.

AMS subject classifications: 76T99, 35J05, 65N06

Key words: Poisson solver, the immersed interface method, two-fluid flow, jump conditions.

1 Introduction

The schematics of an incompressible immiscible two-fluid system is shown in Fig. 1. The two-fluid interface is denoted as $\mathcal{S}$, and its Cartesian coordinates are denoted as $\vec{X}$, as shown in Fig. 1. A single set of conservation equations governing the two-fluid flow reads [11]

$$\rho\left(\frac{\partial\vec{u}}{\partial t}+\nabla\cdot(\vec{u}\vec{u})\right)=-\nabla p+\mu\Delta\vec{u}+\int_{\mathcal{S}}\vec{F}\delta(\vec{x}-\vec{X})\mathrm{d}\mathcal{S}+\rho\vec{g}, \tag{1.1}$$

$$\nabla\cdot\vec{u}=0, \tag{1.2}$$

where $\vec{u}$ is the velocity, p is the pressure, t is time, $\vec{x}$ is Cartesian coordinates, $\vec{F}$ is a force representing interfacial effect, $\delta(\vec{x}-\vec{X})$ is a 3D Dirac delta function, and $\vec{g}$ is a finite

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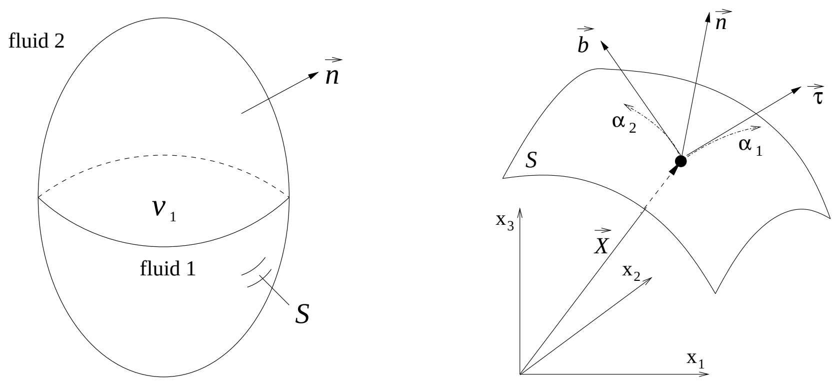

Figure 1: Left: Schematics of a two-fluid system. $\mathcal{V}_1$ is the volume of fluid 1, $\mathcal{S}$ is the two-fluid interface, and $\vec{n}$ is the normal of the interface pointing toward fluid 2; Right: The two-fluid interface $\mathcal{S}$ in a Cartesian coordinate system. $\vec{x}$ is Cartesian coordinates, α_1 and α_2 are two parameters parameterizing the interface locally, $\vec{X}$ is the Cartesian coordinates of the interface, and $\vec{\tau}$, $\vec{b}$ and $\vec{n}$ are unit tangents and normal.

smooth body force. The density ρ and viscosity μ are given by

$$\rho=\rho_1 H(\vec{x},t)+\rho_2(1-H(\vec{x},t)), \tag{1.3}$$

$$\mu=\mu_1 H(\vec{x},t)+\mu_2(1-H(\vec{x},t)), \tag{1.4}$$

where ρ_1 and μ_1 are the constant density and viscosity of fluid 1, ρ_2 and μ_2 are the constant density and viscosity of fluid 2, and $H(\vec{x},t)$ is a 3D step function (Heaviside function) which satisfies

$$H(\vec{x},t)=\begin{cases} 1, & \vec{x}\in\mathcal{V}_1, \\ 0, & \vec{x}\notin\mathcal{V}_1, \end{cases} \tag{1.5}$$

where $\mathcal{V}_1$ is the volume occupied by fluid 1 at time t, as shown in Fig. 1.

By taking the divergence of Eq. (1.1) and applying Eq. (1.2), we obtain a Poisson equation for the pressure p in the two-fluid flow. Away from the two-fluid interface, the Poisson equation reads

$$\Delta p=\rho\nabla\cdot(-\nabla\cdot(\vec{u}\vec{u})+\vec{g}). \tag{1.6}$$

Across the two-fluid interface, the pressure satisfies various jump conditions. In [16], we have derived the following principal jump conditions

$$[p]=\vec{F}\cdot\vec{n}-2[\mu]\left(\frac{\partial\vec{U}}{\partial\tau}\cdot\vec{\tau}+\frac{\partial\vec{U}}{\partial b}\cdot\vec{b}\right), \tag{1.7}$$

$$\left[\frac{1}{\rho}\frac{\partial p}{\partial n}\right]=\frac{\partial}{\partial\tau}\left(\left[\frac{\mu}{\rho}\right]\frac{\partial\vec{U}}{\partial\tau}\cdot\vec{n}-\vec{\tau}\cdot\left[\frac{\mu}{\rho}\frac{\partial\vec{u}}{\partial n}\right]\right)+\frac{\partial}{\partial b}\left(\left[\frac{\mu}{\rho}\right]\frac{\partial\vec{U}}{\partial b}\cdot\vec{n}-\vec{b}\cdot\left[\frac{\mu}{\rho}\frac{\partial\vec{u}}{\partial n}\right]\right), \tag{1.8}$$

$$[\Delta p]=[f], \tag{1.9}$$