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Stochastic Multi-Symplectic Integrator for Stochastic Nonlinear Schrödinger Equation

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Abstract. In this paper we propose stochastic multi-symplectic conservation law for stochastic Hamiltonian partial differential equations, and develop a stochastic multi-symplectic method for numerically solving a kind of stochastic nonlinear Schrödinger equations. It is shown that the stochastic multi-symplectic method preserves the multi-symplectic structure, the discrete charge conservation law, and deduces the recurrence relation of the discrete energy. Numerical experiments are performed to verify the good behaviors of the stochastic multi-symplectic method in cases of both solitary wave and collision.

AMS subject classifications: 60H15, 65M06, 65P10

Key words: Stochastic nonlinear Schrödinger equations, stochastic multi-symplectic Hamiltonian systems, multi-symplectic integrators.

1 Introduction

As is well known, for deterministic Hamiltonian systems, the symplectic integrators for ODEs (see [7,11]), and multi-symplectic integrators for PDEs ([6,8,14]) have been investigated in the last decades, including lots of analysis on accuracy, efficiency, and long-time

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behavior, besides the character of preserving the symplectic or multi-symplectic geometric structure. For stochastic Hamiltonian systems, [15] established the theory about the stochastic symplectic methods which preserve the symplectic structure of the stochastic ODEs. To our knowledge, however, there is no reference about the multi-symplectic structure of stochastic PDEs till now. This motivates us to investigate stochastic PDEs with such structure, and find stochastic multi-symplectic integrators for this kind of stochastic PDEs. We take the stochastic nonlinear Schrödinger equations as the keystone mainly because they describe many physical phenomena and play an important role in fluid dynamics, nonlinear optics, plasma physics, etc [9, 17]. The noise sources usually represent the effect of the neglected terms yields to nonlinear Schrödinger equation in the modelison. A lot of qualitative characteristic for such small noises are presented in several chapters of references [13], including nonlinear-Schrödinger, Korteweg-de Vries, Sine-Gordon equations, etc. The stochastic nonlinear Schrödinger equation can be considered as a generalization of the deterministic nonlinear Schrödinger equation, or from another point of view a perturbation of them. In [5], a perturbed inverse scattering transform technique is used to study nonlinear Schrödinger equation with random terms. If the noise is a space independent case, a transformation can be used to convert the stochastic equation into corresponding deterministic case. Suppose some smooth conditions, the stochastic nonlinear Schrödinger equation has a unique global solution for some cases. For multiplicative noise, [1,16] studied and described some theoretical analysis. For more details about the theoretical aspects of stochastic nonlinear Schrödinger equations refer to [2] and references therein.

This paper is organized as follows. In the next section, we define the stochastic multi-symplectic PDEs, with proof of their preservation of the stochastic multi-symplectic conservation law, and finally give the definition of the stochastic multi-symplectic integrators which preserve the discrete stochastic multi-symplectic conservation law. The stochastic multi-symplectic form for the stochastic nonlinear Schrödinger equation with multiplicative noise is presented, which possesses the charge conservation law. The midpoint method is then used to construct the stochastic multi-symplectic integrator, the concrete form of which for the given Schrödinger equation is obtained by introducing the exact mathematical definition of the space-time noise. Section 3 is contributed to the theoretical analysis of the conservation properties of the obtained stochastic multi-symplectic scheme, including the discrete multi-symplectic conservation law, and the discrete charge conservation law. Furthermore, the recursion formula of a specific energy conservation law is presented. In section 4, numerical experiments are performed to testify the effectiveness of the stochastic multi-symplectic scheme listed in the previous section, for the case of both solitary wave and the collision of solitons. Section 5 is a conclusion.

2 Stochastic NLS equation and multi-symplectic integrator

In this paper, we consider a stochastic nonlinear Schrödinger equation with multiplicative noise: