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A Constructive Method for Computing Generalized Manley-Rowe Constants of Motion

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Abstract. The Manley-Rowe constants of motion (MRC) are conservation laws written out for a dynamical system describing the time evolution of the amplitudes in resonant triad. In this paper we extend the concept of MRC to resonance clusters of any form yielding generalized Manley-Rowe constants (gMRC) and give a constructive method how to compute them. We also give details of a *Mathematica* implementation of this method. While MRC provide integrability of the underlying dynamical system, gMRC generally do not but may be used for qualitative and numerical study of dynamical systems describing generic resonance clusters.

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1 Introduction

Weakly nonlinear wave systems showing resonances can be found in a great variety of physical systems, among them surface water waves, atmospheric planetary waves, plasma drift waves, etc. In Hamiltonian formulation the equation of motion in Fourier space can be written out as

$$i\dot{B}_{\mathbf{k}} = \partial\mathcal{H}/\partial B^*_{\mathbf{k}}, \tag{1.1}$$

where $B_{\mathbf{k}}$ is the amplitude of the Fourier mode corresponding to wavevector $\mathbf{k}$ and the Hamiltonian $\mathcal{H}$ is represented as an expansion in powers of terms $\mathcal{H}_j$, each representing all products of j amplitudes $B_{\mathbf{k}}$:

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_{int}, \quad \mathcal{H}_{int} = \mathcal{H}_3 + \mathcal{H}_4 + \mathcal{H}_5 + \cdots. \tag{1.2}$$

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The quadratic Hamiltonian $\mathcal{H}_2$ describes linear motion and the interaction Hamiltonian $\mathcal{H}_{int}$ describes nonlinear interaction of waves.

If the cubic Hamiltonian $\mathcal{H}_3 \neq 0$, $\mathcal{H}_{int} \approx \mathcal{H}_3$, three-wave interaction is dominant and the main contribution to the nonlinear evolution comes from triads of waves each satisfying the resonance conditions

$$\omega(\mathbf{k}_1)+\omega(\mathbf{k}_2)-\omega(\mathbf{k}_3)=0, \quad \mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3=0. \tag{1.3}$$

The corresponding dynamical system describing time evolution of the amplitudes B_j of a triad reads

$$\dot{B}_1 = ZB_2^*B_3, \quad \dot{B}_2 = ZB_1^*B_3, \quad \dot{B}_3 = -ZB_1B_2, \tag{1.4}$$

the interaction coefficient $Z \neq 0$ being determined from a known function over the solutions of (1.3).

System (1.4) possesses three conservation laws, first found in [1] called Manley-Rowe constants (MRC) of motion. For a resonant triad, the MRC may be written as:

$$I_{13} = |B_1|^2+|B_3|^2, \quad I_{23} = |B_2|^2+|B_3|^2, \quad I_{12} = |B_1|^2-|B_2|^2. \tag{1.5}$$

They are linearly dependent, but any two of them form a linearly independent subset, enough to provide the integrability of (1.4), see [2].

In a 3-wave system, a resonant triad is called a primary resonance cluster, and clusters consisting of more than 3 modes (common clusters) may be decomposed into triads having joint modes, [3]. Common clusters describe many real physical phenomena: in nonlinear water wave systems [4], in piezoelectric semiconductors [5], in geophysics [6,7], in optics [8] etc.

In this paper we present a simple constructive method for deducing generalized MRC (gMRC) for a common resonance cluster. The gMRC are polynomials on $|B_j|^2$ with integer coefficients. The method has been developed by EK for the course on nonlinear resonance analysis held at the J. Kepler University, Linz, since 2005. A *Mathematica* code based on this method has been written by Loredana Tec and can be downloaded from [9]. The method has been used afterwards by the author, collaborators and students but its description has not been published. Recent publication [10] where gMRC are studied by far more complicated methods indicates that a simpler algorithm and available program code may be of interest for physicists working in the area of discrete wave turbulence.

2 Computation of gMRC

2.1 Computation of MRC

Let us rewrite (1.4) as

$$\dot{B}_1 = ZB_2^*B_3, \qquad \dot{B}_2 = ZB_1^*B_3, \qquad \dot{B}_3 = -ZB_1B_2, \tag{2.1a}$$

$$\dot{B}_1^* = ZB_2B_3^*, \qquad \dot{B}_2^* = ZB_1B_3^*, \qquad \dot{B}_3^* = -ZB_1^*B_2^*, \tag{2.1b}$$