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Arbitrary-Lagrangian-Eulerian One-Step WENO Finite Volume Schemes on Unstructured Triangular Meshes

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Abstract. In this article we present a new class of high order accurate Arbitrary-Eulerian-Lagrangian (ALE) one-step WENO finite volume schemes for solving nonlinear hyperbolic systems of conservation laws on moving two dimensional unstructured triangular meshes. A WENO reconstruction algorithm is used to achieve high order accuracy in space and a high order one-step time discretization is achieved by using the local space-time Galerkin predictor proposed in [25]. For that purpose, a new element-local weak formulation of the governing PDE is adopted on moving space-time elements. The space-time basis and test functions are obtained considering Lagrange interpolation polynomials passing through a predefined set of nodes. Moreover, a polynomial mapping defined by the same local space-time basis functions as the weak solution of the PDE is used to map the moving physical space-time element onto a space-time reference element. To maintain algorithmic simplicity, the final ALE one-step finite volume scheme uses moving triangular meshes with *straight* edges. This is possible in the ALE framework, which allows a local mesh velocity that is different from the local fluid velocity. We present numerical convergence rates for the schemes presented in this paper up to sixth order of accuracy in space and time and show some classical numerical test problems for the two-dimensional Euler equations of compressible gas dynamics.

AMS subject classifications: 65Mxx, 35Lxx

Key words: Arbitrary Lagrangian-Eulerian, high order reconstruction, WENO, finite volume, local space-time Galerkin predictor, moving unstructured meshes, Euler equations

1 Introduction

In this paper we present a new family of high order accurate Lagrangian-type one-step finite volume schemes for solving nonlinear hyperbolic balance laws, with non stiff al-

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gebraic source term. The main advantage of working in a Lagrangian framework is that material interfaces can be identified and located precisely, since the computational mesh is moving with the local fluid velocity, hence obtaining a more accurate resolution of material interfaces. For this reason a lot of research has been carried out in the last decades in order to develop Lagrangian methods, whose algorithms can start either directly from the conservative quantities such as mass, momentum and total energy [57,68], or from the nonconservative form of the governing equations, as proposed in [6, 9, 80]. Furthermore we can split the existing Lagrangian schemes into two main classes, depending on the location of the physical variables on the mesh: in one case the velocity is defined at the cell interfaces and the other variables at the cell barycenter, hence adopting a *staggered mesh* approach, while in the other case all variables are defined at the cell barycenter, therefore using a *cell-centered* approach.

The equations of Lagrangian gas dynamics have been considered in [60], where several different Godunov-type finite volume schemes have been presented and where a new Roe linearization has been introduced in order to define proper estimates of the maximum signal speeds in HLL-type Riemann solvers in Lagrangian coordinates. A cell-centered Godunov scheme has been proposed by Carré et al. [10] for Lagrangian gas dynamics on general multi-dimensional unstructured meshes. In this case the finite volume scheme is node based and compatible with the mesh displacement. In [21] Després and Mazeran introduce a new formulation of the multidimensional Euler equations in Lagrangian coordinates as a system of conservation laws associated with constraints. Furthermore they propose a way to evolve in a coupled manner both the physical and the geometrical part of the system [22], writing the two-dimensional equations of gas dynamics in Lagrangian coordinates together with the evolution of the geometry as a weakly hyperbolic system of conservation laws. This allows the authors to design a finite volume scheme for the discretization of Lagrangian gas dynamics on moving meshes, based on the symmetrization of the formulation of the physical part. In a recent work Després et al. [18] propose a new method designed for cell-centered Lagrangian schemes, which is translation invariant and suitable for curved meshes. General polygonal grids are also considered by Maire et al. [54–56], who develop a general formalism to derive first and second order cell-centered Lagrangian schemes in multiple space dimensions. By the use of a node-centered solver [56], the authors obtain the time derivatives of the fluxes. The solver may be considered as a multi-dimensional extension of the Generalized Riemann problem methodology introduced by Ben-Artzi and Falcovitz [5], Le Floch et al. [8, 41] and Titarev and Toro [70, 71, 73]. So far, all the above-mentioned schemes are at most second order accurate in space and time.

In order to achieve higher accuracy, Cheng and Shu were the first who introduced a high order essentially non-oscillatory (ENO) reconstruction in Lagrangian schemes [14, 53]. They developed a class of cell centered Lagrangian finite volume schemes for solving the Euler equations, using both Runge-Kutta and Lax-Wendroff-type time stepping to achieve also higher order in time. Furthermore a formalism for symmetry preserving Lagrangian schemes has been proposed by Cheng and Shu, see [15, 16]. The