Commun. Comput. Phys.
doi: 10.4208/cicp.120712.080313a

Vol. **14**, No. 5, pp. 1372-1414
November 2013

Approximation of H(div) with High-Order Optimal Finite Elements for Pyramids, Prisms and Hexahedra

Morgane Bergot[1] and Marc Duruflé[2,*]

[1] *CALVI project team, INRIA Nancy-Grand Est, Strasbourg, France.*
[2] *BACCHUS project team, INRIA Bordeaux Sud-Ouest, Bordeaux, France.*

Received 12 July 2012; Accepted (in revised version) 8 March 2013

Communicated by Qiang Du

Available online 5 July 2013

Abstract. Classical facet elements do not provide an optimal rate of convergence of the numerical solution toward the solution of the exact problem in $H(div)$-norm for general unstructured meshes containing hexahedra and prisms. We propose two new families of high-order elements for hexahedra, triangular prisms and pyramids that recover the optimal convergence. These elements have compatible restrictions with each other, such that they can be used directly on general hybrid meshes. Moreover the $H(div)$ proposed spaces are completing the De Rham diagram with optimal elements previously constructed for H^1 and $H(curl)$ approximation. The obtained pyramidal elements are compared theoretically and numerically with other elements of the literature. Eventually, numerical results demonstrate the efficiency of the finite elements constructed.

AMS subject classifications: 65M60

Key words: Facet elements, high-order finite element, pyramids, H(div) approximation, De Rham diagram.

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1 Introduction

The aim of this article is to build optimal $H(div)$-conforming elements in the same spirit as done in [1,2] for H^1 and $H(curl)$ conforming formulations. Finite elements for $H(div)$ formulations can be used for example to solve Stokes problem like in [3] or Darcy flow equation (see [4]). The other goal of this article is to complete the De Rham diagram for optimal elements introduced in [1,2] of an hybrid mesh (including hexahedra, tetrahedra, triangular prisms and pyramids).

*Corresponding author. *Email addresses:* `morgane.bergot@inria.fr` (M. Bergot), `marc.durufle@inria.fr` (M. Duruflé)

A first family of finite elements for $H(div)$ formulations has been introduced by Nédélec for hexahedra and tetrahedra in [5], a second one being introduced in [6] for hexahedra, tetrahedra and triangular prisms. Nédélec's first family for $H(div)$ approximation is also known as Raviart-Thomas elements [7]. However, when these elements are used on general unstructured meshes (especially hexahedral meshes), the interpolation error $E_{H(div)}$ is not optimal:

$$E_{H(div)} = ||u-\pi u||_{H(div)} = \mathcal{O}(h^{max(0,r-2)}),$$

where the $H(div)$ norm is defined as

$$||u||^2_{H(div)} = ||u||^2_2 + ||div\, u||^2_2,$$

π being a projector from $H(div)$ on the discrete Raviart-Thomas space, r the order of approximation and h the mesh size. This sub-optimal convergence may be very problematic for lowest-order elements and unstructured meshes including highly distorted elements. The sub-optimal convergence is obtained when the elements of the mesh are not tending to affine elements when h tends to 0. Affine elements are elements such that the transformation F linking reference elements (unit cube, unit prism, unit tetrahedron and symmetric pyramid) to elements of the mesh is affine. This case occurs if the element is only made of triangular faces (i.e. for tetrahedra), or when the quadrilateral faces are parallelograms. In [8], some numerical experiments illustrate the lack of consistency of lowest-order elements. The authors propose non-linear functions in order to obtain a convergent method. An another approach based on the splitting of an hexahedron in tetrahedra is proposed in [4]. These two contributions seem difficult to extend to higher orders and the expression of basis functions on the reference element depend on the geometry.

The aim of this paper is to construct high order elements for hexahedra, tetrahedra, prisms and pyramids ensuring an optimal estimate $\mathcal{O}(h^r)$ in $H(div)$ norm. Such elements will be based on Piola transform, i.e. a basis function u on the real element will be obtained as:

$$u \circ F(\hat{x},\hat{y},\hat{z}) = \frac{1}{|DF|} DF\, \hat{u}(\hat{x},\hat{y},\hat{z}),$$

where $(\hat{x},\hat{y},\hat{z})$ are the coordinates defined on the reference element $\hat{K}$, DF is the jacobian matrix of transformation F (which transforms $\hat{K}$ to the real element K), $|DF|$ its determinant. Moreover, these elements will be constructed such that basis functions on the reference element $\hat{u}$ will belong to finite element spaces $\hat{P}_r$ that do not depend on the geometry, that is to say they do not depend on the real element K. Such finite elements have been obtained for general quadrilateral elements in [9], and for lowest-order hexahedral elements in [10]. In Theorem 2.3, these super-optimal spaces are detailed at any order for the four types of elements, the lowest order hexahedral space is the same as in [10]. However, these spaces are not unique and not very practical to implement, optimal spaces